

Contents

Copyright	ii
Acknowledgements	iv
Abstract	vi
Contents	viii
1. Introduction	1
I. Wittgenstein's Philosophy of Mathematics during the Middle Period	5
A. Wittgenstein	5
B. Wittgenstein and Mathematics	8
C. The Middle Period	13
II. Language, Grammar and Mathematics	17
2. On Mathematical Objects and Concepts	20
I. Introduction	20
II. Wittgenstein's Criticism of the Concept-Object Distinction	22
A. <i>Zahlangaben</i>	22
B. Frege's Grammatical Enquiry into the Concept of Number	25
C. The Grammatical Method	27
D. Wittgenstein's Criticism of Frege's Grammatical Analysis of <i>Zahlangaben</i>	28
E. Mathematical Objects and Concepts	31

III. On Mathematical Equations	35
A. Frege and Ramsey: Mathematical Equations as Identity Statements	35
B. Equations and Calculations	41
IV. An Extension of Frege’s Context Principle	43
A. The Context Principle	43
B. Specification and Calculation Propositions and Concepts	43
V. Beyond Arithmetic, Proofs as Calculations	45
VI. Conclusion	48
3. The Grammar of Calculation	50
I. Introduction	50
II. Mathematics as Calculation	51
A. “There’s a <i>Fascination</i> here:” Mathematical Problems and Problems of Mathematical Investigation	51
B. Method and Solution	55
C. Calculations are Transitions between Expressions According to a Rule	56
III. Mathematical Correctness	58
A. Wittgenstein’s Anti-Platonism	59
1. Platonist Mathematical Explorations	59
2. Searches and Explorations	62
B. Correct Calculations and Correct Results	66
1. Calculations as Transitions	66
2. Traces	68
IV. Conclusion	70

4. Mathematical Application [<i>Anwendung</i>]	72
I. Introduction	72
II. Calculation's Role in the Solution of <i>Practical</i> Problems	75
A. Wittgenstein's Criticism of the Logicians' Account of Mathematical Application	77
1. The Logician's Puzzle	77
2. First Myth: Mathematics is the Most Universal of Sciences	78
3. Wittgenstein against the Universality of Mathematics	79
4. Letters and Schemes	80
5. Wittgenstein and Ramsey on <i>Anwendung</i> and Interpretation	84
B. Wittgenstein's Account of Mathematical Application	87
III. <i>Anwendung</i> and the Foundations of Mathematics	91
A. The Autonomy of Mathematical Calculi	91
B. <i>Anwendung</i> is an Essential Feature of Mathematics	94
C. On Consistency [<i>Widerpruchsfreiheit</i>]	97
IV. Conclusion	99
5. Grammar	101
I. Introduction	101
II. A Formal Background for the Discussion of Wittgenstein's Grammar	102
A. Language	103
B. What is Grammar?	104
C. The Conventional Approach	108
D. Example: The Language of Propositional Calculus	114
1. The Language of Propositional Calculus	114

2. The Grammar of Propositional Calculus	114
E. Strong, Redundant and Trivial Grammars	119
1. Strong Grammars	120
2. Redundant Grammars	121
3. Trivial Grammars	124
III. Wittgenstein's Approach	124
IV. A comparison between the Two Approaches	126
A. Wittgenstein Grammar	126
B. Formal Requirements for a Wittgenstein Grammar	131
V. Conclusion	135
6. Mathematics as Grammar. A Formal Treatment	136
I. Introduction	136
A. A Formal Model of Grammar	136
B. <i>Anwendung</i> and the Grammar of Natural Language	137
C. Mathematics as Grammar	138
II. A Grammatical Analysis of the Internal <i>Anwendung</i> of Arithmetic	140
A. Introduction	140
1. Calculi as Languages	140
2. A Grammatical System of Natural Numbers	140
3. The Grammar of Addition	141
B. A Formal Grammatical Analysis of a Subsystem of the Arithmetic of Natural Numbers with Addition as its Single Operation	142
1. The Calculus as Language	142
2. Grammatical Number Theory	143
3. The Grammar of Addition	143

C. Mathematical Induction	148
III. A Grammatical Analysis of the External Application of Arithmetic	149
A. A Numerical System	150
1. ‘One’	150
2. ‘Two’	155
3. From ‘One’ to ‘Two’ and Beyond: Succession and Induction	155
B. Calculation: the Case of Division	160
IV. Conclusion: A False Dilemma	161
7. <i>Zahlangaben</i> Revisited	164
I. Introduction	164
II. Grammatical Concepts and Systems of Propositions	164
A. Systems of Propositions	164
1. Systems of Propositions Formally Defined	168
2. Complete Descriptions	170
3. Grammatical Concepts Formally Defined	171
B. Grammatical Concepts and Systems of Propositions	172
1. A Warning Notice	174
III. <i>Zahlangaben</i> Revisited	176
A. On the Extensionality of Specification Concepts	177
B. Arithmetical Propositions	181
IV. Conclusion	183
8. Grammatical Necessity	185
I. Introduction	185
II. On the Grammatical Nature of Mathematics	187

A. Arguments against the Grammatical Nature of Mathematics	187
B. Grammatical Statements	189
C. Grammatical Propositions as Rules	194
D. Grammar and Vocabulary	196
III. Wittgenstein's Syntactic Necessity and Analyticity	199
A. Brief Historical Background	199
B. Carnap	201
C. Quine	205
1. "Two Dogmas of Empiricism" and the Analytic Nature of Grammar	205
2. Convention and Justification	209
IV. Conclusion: Wittgenstein's Own Account of Analyticity	212
9. Bibliography	215

Mathematics as Grammar

'Grammar' in Wittgenstein's
Philosophy of Mathematics
during the Middle Period

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May this humble work
serve to glorify the Lord.

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1. Furthermore, her grammatical and stylistic corrections served as informal evidence to the main hypothesis of this dissertation. Her corrections included not only obviously grammatical rules, but also transformation rules with a natural logical interpretation, such as DeMorgan rules or exchange of quantifiers.

Abstract

This dissertation looks to make sense of the role ‘grammar’ plays in Wittgenstein's philosophy of mathematics during the middle period of his career. It constructs a formal model of Wittgenstein's notion of *grammar* as expressed in his writings of the early thirties, addresses the appropriateness of that model and then employs it to test Wittgenstein's claim that mathematical propositions are ultimately grammatical.

In order to test Wittgenstein's claim that mathematical propositions are grammatical, the dissertation provides a formalized theory of grammatical analysis and applies it to a portion of language big enough to contain numerical expressions. It attempts to prove that, if the object language contains the appropriate numerical expressions, the resulting grammar will include at least some rules with a natural mathematical interpretation. In particular, it tries to show that Wittgenstein's grammatical analysis of the ordinary use of numerical expressions yields familiar theorems of arithmetic.

The dissertation also endeavors to extend these results to a coherent picture of Wittgenstein's peculiar view of mathematics as grammar. It fits the formal results into the larger picture of Wittgenstein's philosophy of mathematics during this period. A large portion of the dissertation explains how, by combining the notions of grammar and mathematics, Wittgenstein allowed himself to offer original answers to some central questions in the philosophy of mathematics. In this regard, the dissertation pays special

attention to four main issues: (i) Wittgenstein's use of the term 'grammar' in his philosophical writings of this period, in comparison with traditional understandings of 'grammar', (ii) mathematics as part of the syntax of language, (iii) his explanation of mathematical application, and (iv) Wittgenstein's account of mathematical necessity. Taking these points in consideration situates the dissertation's results in the larger philosophical discussion of these themes.

Chapter 1

Introduction

In the early 1930s, Ludwig Wittgenstein advanced the thesis that mathematical statements are ultimately grammatical. Most of Wittgenstein's audience at the time were philosophers and mathematicians with little or no knowledge of linguistics. They found the idea that mathematics could be part of grammar – what they called “the dullest of school subjects” – inconceivable. Consequently, they interpreted ‘grammar’ as some esoteric logical syntax of scientific language.¹ Despite the profound development of grammatical studies in the 20th century, many philosopher's prevalent attitude towards grammar has changed little. The time has come for a theoretical reevaluation of Wittgenstein's thesis in light of the most recent developments in theoretical grammar. This dissertation explains grammar's role in Wittgenstein's philosophy of mathematics during the early thirties. It answers two central questions, ‘Can mathematical propositions be grammatical?’ and ‘How does Wittgenstein support this thesis?’ It explains Wittgenstein's claim that mathematical propositions are ultimately grammatical and describes this claim's role in his philosophy of mathematics during that period.

¹. In his “Intellectual Autobiography,” Carnap writes about the Vienna Circle, “When we found in Wittgenstein's book [The *Tractatus Logico Philosophicus*] statements about “the language,” we interpreted them as referring to an ideal language; and this meant for us a formalized symbolic language. Later Wittgenstein explicitly rejected this view.” Paul Arthur Schilpp, ed. *The Philosophy of Rudolf Carnap*, The Library of Living Philosophers, vol. XI (La Salle: Open Court, 1963) 29.

Philosophers have questioned whether or not mathematics is part of the formal grammar of language for more than a century. This question lies at the center of the debate between Carnap and Bar-Hillel, on the one side, and Gödel, Tarski and Quine on the other. Carnap asserted that his philosophy of mathematics as syntax originated in Wittgenstein. In consequence, it is important to clarify whether or not Wittgenstein held a view similar to the one Carnap championed and, if so, to defend him against Quine's criticisms.

Throughout the *Big Typescript*, *Philosophical Grammar* and *Philosophical Remarks*, Wittgenstein developed most of his ideas on the philosophy of mathematics through examples from elementary arithmetic. Following his own presentation, this dissertation concentrates on the case of mathematical numerical expressions and their calculi. However, since Wittgenstein's notion of calculation includes mathematical processes like drawing geometrical figures, and proving theorems within a formal system, his results rightfully extend over all mathematics. On this topic, S. G. Shanker writes at the beginning of *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*,

The more [Wittgenstein] addressed the fundamental confusions underlying the 'foundations crisis' the more strongly he began to feel that the philosophical problems which surface in the various realms of higher mathematics are merely more complex versions of the same issues which arise in elementary arithmetic. For example, the type of problems that emerged with the construction of the transfinite cardinals are essentially the same as those that characterize the construction of any new number system. Hence, Wittgenstein sought to gain in perspicuity what he lost in detailed application by presenting his criticisms of the questions which prefigure in higher mathematics in the context of the problems which occur in elementary arithmetic.²

The first chapter develops some preliminary notions in Wittgenstein's philosophy of mathematics. It introduces the notions of grammatical concept and object. Wittgenstein bases his argument on these two notions.

2. S. G. Shanker, *Wittgenstein and the Turning-Point in the Philosophy of Mathematics* (Albany: State University of New York Press, 1987), 5.

The second chapter develops Wittgenstein's idea that mathematical propositions connect calculations with their final results. For example, the arithmetical proposition ' $3 + 4 = 7$ ' says that adding three to four results in seven. In developing this idea, Wittgenstein criticizes alternative philosophical approaches to mathematical numerical expressions [*Zahlangaben*] – in particular, Frege's and Ramsey's accounts of arithmetical equations, Frege's seminal work on the concept of number, and Russell's notion of cardinality.

The third chapter explains why mathematical propositions are calculation rules. Calculation is a rule-governed linguistic practice and calculi are grammatical systems. Every mathematical calculus is a linguistic system with its own grammar. This grammar determines correct or incorrect calculations. The calculation's result is correct if the calculation itself is correct, not vice versa. Wittgenstein opposed the Platonists' view of calculations as *expeditions* into uncharted mathematical territory. Instead, he envisions calculations as *searches* over well defined grammatical spaces.

Wittgenstein's claim that mathematical propositions are grammatical also means that they are part of the grammar of natural language. Understanding mathematics' relationship to natural language requires analyzing mathematical application [*Anwendung*]. This analysis takes place in the third chapter.

From the point of view of calculation, pure and applied mathematics are not significantly different. Pure and applied mathematics consist entirely of calculations [*Rechnungen*]. The grammar of mathematical expressions is the same in natural language and in pure mathematics. The calculus determines the grammar of mathematical expressions in natural language as well as in calculation. In consequence, mathematical calculi are part of the grammar of natural language. Mathematics are applicable only inside some defined calculus [*Kalkül*]. Calculation says nothing about matters outside the calculus. Mathematical

calculations help solve non-mathematical problems. They provide the grammar of non-mathematical hypothesis. However, they do not justify or entail the hypothesis' truth.

The fourth chapter explains Wittgenstein's use of 'grammar'. It demonstrates formally that numerical calculi form grammatical systems. It also provides a formalized theory of grammatical analysis. The fifth chapter applies this theory to prove that mathematical calculi are part of natural language grammar. They are part of the grammar of the segment of natural language in which they occur. If the object language contains the appropriate mathematical expressions, the resulting grammar includes at least some rules with a natural mathematical interpretation. In particular, the grammatical analysis of numerical expressions in natural language obtains familiar arithmetical axioms.

Using a formal method to study Wittgenstein's philosophy is highly controversial. He explicitly opposed formal methods in philosophy.³ However, good reasons support adopting a formal method. Working with the notion *grammatical* in a formal context is essential to clarify Wittgenstein's claim that mathematical propositions are grammatical. The formal approach provides a rigorous understanding of the adjective 'grammatical'. However, the ultimate subject of this formal reconstruction and analysis is Wittgenstein's philosophy of mathematics. Even though some of this analysis' formal results might have importance of their own, formal logic plays only a supporting role.

Finally, the seventh chapter addresses the Carnap-Gödel debate on the syntactic nature of mathematics. It elaborates on notions developed in previous chapters to explain the analytic nature of mathematical propositions. It develops Wittgenstein's account of syntactic necessity, and defends it from Quine's arguments.

³. "He [Wittgenstein] had a skeptical and sometimes even a negative view of the importance of a symbolic language for the clarification and correction of the confusions in ordinary language. . ." (Carnap 1963, 29)

I. Wittgenstein's Philosophy of Mathematics during the Middle Period

A. Wittgenstein

Wittgenstein's philosophical ideas, not to mention his unconventional means for expressing them, are so radical that interpreters cannot arrive at a consensus on how to approach his work or decide upon its lasting significance. He figures prominently among the early masters of so-called analytical philosophy, like Frege, Russell, the Vienna circle and the Oxford school of ordinary language philosophy. His name frequently appears in connection with thinkers as diverse as Shopenhauer, Kirkegaard, Heidegger or Derrida. Authors allude to his work even in topics like gardening, rhetoric, mysticism, architecture, and deep psychology.

The complex historical conditions of his life and thought make him an eminent figure in the intellectual history of the 20th Century. He was fortunate enough to come of age in *fin-de-siècle* Vienna, at the same time than such central figures of early 20th Century Western History as Gustav Mahler, Sigmund Freud, Gustav Klimt and Adolf Loos. A quarter of a century later, he partook in another intellectual revolution, when the Vienna Circle drew him into its deliberations. His name is just as much connected with the intellectual history of his hometown as it is with that of Cambridge. Upon his first arrival in Cambridge, Wittgenstein found himself surrounded with some of the leading English intellectuals of the period: Bertrand Russell, Alfred North Whitehead, John Maynard Keynes and Lytton Strachey. During the thirties and early forties, he again became part of a strong academic community featuring G. E. Moore, and Pierro Straffa. As a professor, Wittgenstein also enjoyed a following of students like Norman Malcolm, Rush Rhees and Elizabeth Anscombe, who helped spread his ideas throughout the English-speaking world.

Despite the continuous effort of some interpreters to frame his iconoclastic thinking within a philosophical tradition or doctrine, Wittgenstein did not even adhere to his own

doctrine. In *A Biographical Sketch*, composed not long after Wittgenstein's death, G. H. von Wright wrote:

Wittgenstein [partly] repudiated the results of his own influence. He did not participate in the worldwide discussion to which his work and thought had given rise. He was of the opinion --justified I believe-- that his ideas were usually misunderstood and distorted even by those who professed to be his disciples. He doubted that he would be better understood in the future. He once said that he felt as though he were writing for people who would think in a quite different way, breathe a different air of life, from that of present-day men.⁴

Wittgenstein was an extreme example of a Mexican '*mamón*'. His intense personality elicited extreme responses in the few who met him in person. People admired and feared him. Most of all, his extraordinary intelligence and aloofness charmed them. When asked to write an assessment of Wittgenstein for a symposium, Norman Malcolm wrote in 1960:

Wittgenstein's conversation made an overwhelming impression because of the united seriousness and vivacity of his ideas, and also because of the expressive mobility of his beautiful face, the piercing eyes and commanding glance, the energetic movements and gestures. In comparison, someone has remarked, other people seemed only half alive.⁵

He refused to associate with those he found undesirable. Some say that Wittgenstein avoided making acquaintances, but needed and sought friendships.⁶ According to Jaakko Hintikka, Wittgenstein "was a philosophical genius, but he was socially and intellectually a lone wolf who did not assume any responsibility for, or even exhibit an interest in, many of the institutions of our society and culture."⁷ Although he tried to avoid publicity, his reclusive behavior prompted the growth and dissemination of numerous legends about his perso-

4. G. H. von Wright, "Ludwig Wittgenstein: A Biographical Sketch" in Norman Malcolm ed. *Wittgenstein* (Minneapolis: University of Minnesota, 1960), 15.

5. *Ibid.*

6. *Ibid.*, 31.

7. Jaakko Hintikka, "Who is about to Kill Analytic Philosophy?" in Anat Biletzki and Anat Matar, *The Story of Analytic Philosophy* (New York: Routledge, 1998), 259.

nality. As a result, his name appears today in detective novels and art films almost as often as in history books and philosophy journals.

In contrast with the pervasive influence of his ideas, Wittgenstein remains an elusive philosopher. Because he resisted theorizing, wrote in an aphoristic style, and expressed his thoughts in a highly personal, even existential tone, Wittgenstein did not fit comfortably in academia.⁸ Although he taught in Cambridge for more than a decade, Wittgenstein disdained the scholastic pedantry of the academic philosophy of his days. On the contrary, he continually challenged the protocols of academia both in his writings and professional practice. In a personal description of Wittgenstein, Gilbert Ryle wrote:

He loathed being connected with academic philosophers, and he avoided academic chores. After 1929 he attended no conferences; he did no reviewing for journals; only once did he attend a philosophical meeting in Oxford; he was inaccessible to visiting philosophers; he read few, if any, of the philosophical books and articles that came out during his last 25 years.⁹

Paradoxically, academia could not have more thoroughly embraced another philosophical figure in the 20th century. Every year, the study of his philosophical ideas fills numerous articles, books, dissertations and journals. Nevertheless, much of his rich and complex work remains unexplored.

⁸. Some interpreters have argued that a methodical reading of Wittgenstein's texts does not adequately match the disjoint quality of his writing. However, this appraisal of Wittgenstein's thought is unjustifiably condescending.

⁹. Gilbert Ryle, "The Work of an Influential but Little-known Philosopher of Science: Ludwig Wittgenstein" (Shanker 1986, 138)

B. Wittgenstein and Mathematics

Die philosophische Klarheit wird auf das Wachstum der Mathematik den gleichen Einfluß haben, wie das Sonnenlicht auf das Wachsen der Kartoffeltriebe. (In dunkeln Keller wachsen sie meterlang.)

PG §25 p.750

Philosophical clarity will have the same effect on the growth of mathematics as sunlight has on the growth of potato shoots. (In a dark cellar they grow meters long.)

PG §25 p. 381

Wittgenstein's philosophical writings on mathematics call into question the boundaries between philosophy and mathematics. His work reveals the philosophical issues behind some problems otherwise considered as purely mathematical. For example, he found that neither logic nor any other calculus could serve as a foundation for arithmetics. For this reason some mathematicians have accused Wittgenstein of technical incompetence. However, when trying to expose Wittgenstein's mathematical errors, these mathematicians have found themselves dealing with deep philosophical issues. In his philosophical writings on mathematics, Wittgenstein did not blur the line between mathematics and philosophy, but challenged the traditional way of separating them.

On entering the subject of mathematics, some authors find it proper to state that Wittgenstein disclaimed any specialist knowledge of mathematics. For example, Jaakko Hintikka writes,

Wittgenstein had no sympathy for, or real understanding of, mathematical . . . theorizing. For all his aesthetic sensibilities, he had for instance no feeling for the elegance and power of a real mathematical theory. There are no indications that he had any appreciation of, or even knowledge of, such things as Galois theory, the calculus of residues, Gauss-Riemann surface theory, or the theory of Hilbert spaces.¹⁰

“For Wittgenstein has been accused,” remarks S. G. Shanker, “not simply of subversive – even anarchic – tendencies, but even worse, he has repeatedly been charged with that most

¹⁰. (Hintikka 1998, 259)

heinous of crimes: technical incompetence.”¹¹ However, his knowledge of mathematics did not derive “from extensive reading, but from a working familiarity with mathematical techniques.”¹² Elementary arithmetic covers a big portion of Wittgenstein’s *Remarks on the Foundations of Mathematics*. However, this is not evidence of his technical incompetence. His manuscripts of the thirties contain meticulous and detailed analyses of a wide range of non-elementary mathematical issues: from Cantorean transfinite number theory and the continuum problem to Skolem’s recursive proof and Hilbert’s various attempts to construct a consistency proof.¹³

Through the many changes in Wittgenstein’s life and thought, few concerns remained of greater importance than the foundations of mathematics. As early as his late teenage years, Wittgenstein had already developed an “amateur’s fascination” with mathematics. His interest in the philosophical aspects of mathematics began during his days at Manchester University. After earning his certificate in engineering at the *Technische Hochschule* in Berlin-Charlottenburg, Wittgenstein registered in the fall of 1908 at Manchester as an engineering research student. At this time, discussions with Horace Lamb and lectures from J. E. Littlewood led him to read Russell’s recently published *The Principles of Mathematics*. Eventually, his growing interest in mathematical logic and the philosophy of mathematics led him to abandon engineering and to focus on these subjects. After leaving Manchester, Wittgenstein visited Gottlob Frege, who advised him to study logic with Russell at Cambridge University. Once there, Wittgenstein’s philosophical vocation extended beyond the limits of logic and mathematics. From that moment on, he became vitally immersed in philosophy of language, metaphysics and, later, mysticism, ethics and aesthetics as well.

¹¹. S. G. Shanker, “Introduction: The Portals of Discovery” (Shanker 1986, 1)

¹². (von Wright 1960, 33) Nevertheless, this is an understatement. Wittgenstein kept up with contemporary developments in mathematics through extensive reading.

¹³. (Shanker 1986, 3)

Wittgenstein's attitude towards these issues was not that of detachment, but vigorous engagement. He found the aloofness of academia increasingly unbearable. So much that, when World War I erupted, he abandoned Cambridge.

According to most accounts, Brouwer's lecture "Mathematics, Science and Language" [*Mathematik, Wissenschaft und Sprache*], which Wittgenstein attended in Vienna in 1928, moved him to resume philosophy. Wittgenstein did not leave the lecture converted to Brouwer's intuitionistic project, but something in that lecture struck a responsive chord in him. As a matter of fact, he focused on the philosophy of mathematics, figuring prominently in his typescripts, manuscripts and lectures from the period. According to P. M. S. Shanker, "Approximately one-third of the *Big Typescript* is concerned with the philosophy of mathematics; indeed, it should not be forgotten that in the years between 1929 and 1944 about half of Wittgenstein's writings were on his subject"¹⁴.

Philosophy of mathematics remained central to Wittgenstein's thought during the later period of his life. A broad selection of his remarks on this topic written in the years 1937-1944 were published in 1956 as *Remarks on the Foundations of Mathematics*. Wittgenstein's originally intended to incorporate these remarks to the *Philosophical Investigations*.¹⁵ In fact, Part I of the *Remarks* was part of the first version of the *Investigations* written in Norway during 1937. After being elected to hold Moore's chair at Cambridge, Wittgenstein delivered a series of lectures on the philosophy of mathematics. These lectures were later published as *Wittgenstein's Lectures on the Foundations of Mathematics*. However, these reflections are not the last chapter in Wittgenstein's life-long liaison with mathematics and its philosophical problems. Almost until his final days, he continued writing on these topics. Rush Rhees remembers that when John Wisdom asked Wittgen-

¹⁴. Ibid. 87

¹⁵. G. H. von Wright, R. H. Rhees and G. E. M. Anscombe, editor's preface to Ludwig Wittgenstein. *Remarks on the Foundations of Mathematics* (Cambridge: MIT Press, 1956), vi.

stein in 1944 to suggest a dictionary entry about his philosophy, he wrote just one sentence: “He has concerned himself principally with questions about the foundations of mathematics.”¹⁶

Despite his extreme enthusiasm for mathematics and its philosophy, Wittgenstein eliminated most remarks on the foundations of mathematics from the published version of the *Philosophical Investigations*. In consequence, his ideas on this subject remained largely unknown until the posthumous publication of his manuscripts, typescripts and lecture notes. Wittgenstein’s first posthumous volume on mathematics was the *Remarks on the Foundations of Mathematics*, from 1956. Unfortunately, the philosophical community did not react encouragingly. In his introduction to the third volume of *Ludwig Wittgenstein, Critical Assessments*, Stuart Shanker writes about “a storm of calumny . . . raging” at the aftermath of its publication.¹⁷ Morris Engel writes about “the almost unanimous sense of disappointment and disapproval”¹⁸ in early reviews. Even advocates of Wittgenstein’s later philosophy dismissed it as erratic and misinformed. For example, the opening paragraph from Michael Dummett’s seminal article from 1959, “Wittgenstein’s Philosophy of Mathematics,” says,

From time to time Wittgenstein recorded in separate notebooks thoughts that occurred to him about the philosophy of mathematics. His recently published *Remarks on the Foundations of Mathematics* consists of extracts made by the editors from five of these. Neither it nor any of these notebooks was intended by its author as a book. That it cannot be considered, and ought not to be criticized, as such is therefore unsurprising, though disappointing. Many of the thoughts are expressed in a manner which the author recognized as inaccurate or obscure; some passages contradict others; some are quite inconclusive; some raise objections to ideas which Wittgenstein held or had held which are not themselves stated clearly in the volume; other passages again, particularly those on consistency and on Gödel’s theorem, are of poor quality or contain definite errors.

¹⁶. Michael Nedo, introduction to *Wiener Ausgabe* (New York: Springer-Verlag, 1993), 57.

¹⁷. (Shanker 1986, 1)

¹⁸. S. Morris Engel, “Wittgenstein’s *Foundations* and Its Reception” (Shanker 1986, 146), 257-68.

Dummett also wrote that some of Wittgenstein's remarks are "plainly silly," "extremely hard to swallow," "extraordinarily difficult to take . . . seriously," "thin and unconvincing." Dummett was not alone in his disappointment with the *Remarks on the Foundations of Mathematics*. Georg Kreisel ended his review saying, "It seems to me to be a surprisingly insignificant product of a sparkling mind."¹⁹ Alan Ross Anderson wrote in his review that "it is not hard to reach the conclusion that Wittgenstein failed to understand clearly the problems with which workers in the foundations have been concerned."²⁰ A page later he wrote, "It is very doubtful whether this application of his method to questions in the foundations of mathematics will contribute substantially to his reputation as a philosopher."²¹

Comments of this sort set the tone for almost all further discussion. It came as no surprise, then, that the rest of Wittgenstein's writings on the subject received a similar condescending and dismissive response. Even today, some Wittgenstein scholars allege that most of his remarks on the philosophy of mathematics are largely wrong, and that having a complete picture of Wittgenstein's thought is the only reason to study them.²² In consequence, some of Wittgenstein's least studied writings are from the middle period. In his introduction to the 1996 *Cambridge Companion to Wittgenstein*, Hans Sluga admits that Wittgenstein's remarks from the thirties "on the philosophy of mathematics have remained

19. G. Kreisel, "Wittgenstein's *Remarks on the Foundations of Mathematic*" *British Journal for the Philosophy of Science* 9 (1958-59) : 158 quoted in. Michael Wrigley, "Wittgenstein's Philosophy of Mathematics" (Shanker 1986, 183)

20. Alan Ross Anderson, "Mathematics and the Language Game" *The Review of Mathematic* 11 (1958) : 457 quoted in Morris Engel (Shanker 1986, 147)

21. Ibid. 458.

22. For example, in the preface of his *Wittgenstein on the Foundations of Mathematic* (Harvard: Cambridge, 1980), Chrispin Wright warns that "the easy stance of eclecticism" – to recover what one thinks is right, and dismiss what one may dislike in Wittgenstein – "is not an option".

among Wittgenstein's most controversial and least explored writings.”²³

C. The Middle Period

Wittgenstein's middle period ranges from his return to Cambridge, early in 1929, to 1933.²⁴

According to Brian McGuinness and G. H. Von Wright, Wittgenstein visited Cambridge for a holiday, but he quickly decided to stay.²⁵ Before readmission to Trinity, he stayed with J. M. Keynes at King's College, and later with Lettice and Frank Ramsey. On 18 January 1929, he began as a research student, working towards a Ph.D. degree. However, his status obviously did not correspond to that position. In consequence, the university offered to count his pre-war residence at Cambridge as credit towards the degree and the *Tractatus*, published eight years earlier, as a thesis. Ramsey was formally his supervisor, and Moore and Russell, his examiners. He received his degree on June 18, 1929. During the academic year 1929-30, he lectured on philosophical logic at the invitation of the Moral Sciences Faculty Board. By the end of 1930, he was a fellow of Trinity College. The fellowship extended until the end of the academic year 1935-6, when his Faculty Lectureship also ended.

Wittgenstein initially returned to Cambridge to correct certain difficulties in the

²³. Hans Sluga, introduction to *The Cambridge Companion to Wittgenstein* (New York: Cambridge, 1996), 17.

²⁴. In this point, this dissertation sides with Dale Jacquette. In the introduction to *Wittgenstein's Thought in Transition*, he wrote, “I designate [the middle period] from 1929 to 1933. . . . The dates are significant and by no means arbitrary. . . . In 1930, Wittgenstein began lecturing at Cambridge University. The end of the transition period can be dated approximately to 1933, because Wittgenstein's lectures from this term recorded in the Blue Book, together with the *Brown Book* of 1934, already contain his new methodology and nearly all of the central ideas of his later philosophy as they were to appear in the *Philosophical Investigations*.” (West Lafayette: Purdue University Press 1998), 9.

²⁵. G. H. von Wright & B. McGuinness, introduction to *Ludwig Wittgenstein: Cambridge Letters. Correspondence with Russell, Keynes, Moore, Ramsey and Sraffa* (Oxford: Blackwell, 1995), 4. However, G. E. Moore disagrees. In an introductory passage from “Wittgenstein's Lectures in 1930-1933,” he writes, “The statement in the Obituary notice in *The Times* for May 2, 1951, that he arrived in Cambridge in 1929 “for a short visit” is very far from the truth.” Robert R. Ammerman ed., *Classics of Analytic Philosophy* (New York: McGraw-Hill, 1965), 234 n. 1.

Tractatus Logico-Philosophicus. According to most accounts, a sense that the original project of the *Tractatus* remained incomplete brought him back to academia, not a new philosophical outlook. Nevertheless, as he rethought the *Tractatus*, he realized that nothing less than a radical transformation would do. Consequently, Wittgenstein's philosophical thought evolved rapidly from 1929 to 1933.²⁶

Except for some conversations and correspondence with Ramsey²⁷ and a few members of the Vienna Circle,²⁸ Wittgenstein isolated himself from philosophy at the completion of the *Tractatus*. Nevertheless, with his return to the academic world, Wittgenstein began to write about philosophy again. The tremendous output of the following years in Cambridge produced two bulky typescripts later published under the title *Philosophical Remarks (Philosophische Bemerkungen)*, notebooks composed between

26. (Shanker 1986, 4) Nevertheless, Wittgenstein's thought evolved in a continuous and gradual way. It suffered no sudden radical shifts. Remarkable on the apparently radical differences between Wittgenstein's early and later work, Rush Rhees wrote, "He returned again and again to the question that had occupied him from the beginning. Like anyone else who does this, he came to see difficulties in many of the ideas he had once accepted. In some respects he came to see the problems differently. And as he did so he saw that other methods were needed for the study of them. In all this, I must repeat, he was going more deeply into the problems he had studied at the time when Russell [who had suggested that he throw away his great talent for philosophy in the last twenty years of his life] admired him. If there was anything 'singular' about the changes he made, it was the penetration he showed – the way in which he would recognize difficulties which no one else would have noticed – and in the persistence with which he discussed the same things". Rhees, Rush, "Ludwig Wittgenstein: a Symposium. Assessments of the Man and the Philosopher" (*The Listener*, IV January 28 and February 4, 1960), 208. Republished in John V. Canfield, *The Philosophy of Wittgenstein: Vol. 4: The Later Philosophy – Views and Review* (New York:Garland Publishing, Inc., 1986), 106.

27. Ramsey visited Wittgenstein for the first time in 1923, with the purpose of correcting his English translation of the *Tractatus*. They soon became friends. Their frequent conversations and correspondence inspired much of Wittgenstein's post-*Tractatus* ideas on mathematics. They remained friends until Ramsey's premature death at age 26, in January 1930. (Sluga 1996, 18)

28. In his "Intellectual Autobiography," Carnap writes that, "In 1927 Schlick became personally acquainted with Wittgenstein. Schlick conveyed to him the interest of our Circle [the Vienna Circle] in his book [the *Tractatus Logico-Philosophicus*] and his philosophy and also our urgent wish that he meet with us and explain certain points in his book which had puzzled us. But Wittgenstein was not willing to do this. Schlick had several talks with him; and Wittgenstein finally agreed to meet with Waismann and me. Thus the three of us met several times with Wittgenstein during the summer of 1927. . . I regretted it when he broke off the contact. From the beginning of 1929 on, Wittgenstein wished to meet only with Schlick and Waismann, no longer with me or Feigl, who had also become acquainted with him in the meantime, let alone with the Circle." (Schilpp 1963, 25-27)

February 1929 and July 1930, first published in 1964) and *Philosophical Grammar* (*Philosophische Grammatik*, written between 1932 and 1934, first published in 1974). Despite being virtually finished works, Wittgenstein did not publish either of them. The need to obtain a research fellowship at Trinity College at the end of 1930 forced him to write the typescript now published as *Philosophical Remarks*. Bertrand Russell reported to the Council of Trinity College, which was considering the award,

The theories contained in this new work of Wittgenstein are novel, original, and indubitably important. Whether they are true, I don't know. As a logician, who likes simplicity, I should wish to think that they are not, but from what I have read of them I am quite sure that he ought to have an opportunity to work them out, since when completed they may easily prove to constitute a whole new philosophy.²⁹

Other important sources for the study of Wittgenstein's thought during this period are Moore's and Lee's³⁰ lecture notes from 1930 to 1933, and the *Big Typescript* from 1933. According to Hacker's account of Wittgenstein's life at Cambridge,³¹ the *Big Typescript* is the closest Wittgenstein came to completing a draft during this period. It is 768 pages long, including an annotated table of contents. Nevertheless, as soon as he 'finished' it, he started making additions, deletions and alterations. These corrections continued sporadically until 1937, the date of the first version of the future *Philosophical Investigations*. This typescript is the last document of Wittgenstein's thought from the middle period. 200 of the remarks in the *Investigations* already occur in it or its revisions.³²

Even though Wittgenstein did not himself publish any of his ideas during this period, they are not worthless. These writings remained unpublished during Wittgenstein's lifetime, because "he was never quite content with how he had stated his views or ordered

²⁹. (von Wright 1960, 26 n. 13)

³⁰. Desmond Lee's notes were published as *Wittgenstein's Lectures, Cambridge 1930-1932*.

³¹. P. M. S. Hacker, *Wittgenstein's Place in Twentieth-Century Analytic Philosophy* (Cambridge: Blackwell, 1996), 79.

³². (Shanker 1986, 86)

the remarks in which they were expressed.”³³ On the topic of Wittgenstein’s excessively high standards, Norman Malcolm remembers,

With respect to philosophical work his standards were inexorable. Of a young friend who was preparing a paper to read to the Moral Science Club at Cambridge he remarked that he ought to write it for a hundred years from now and not just for next week. This he said of a paper that was intended merely for a discussion group, not for publication.³⁴

Unfortunately, Wittgenstein’s philosophy from this period remains relatively unstudied. Testimony suggests that Wittgenstein himself dismissed any other stage in the development of his philosophy besides those of the *Tractatus* and the *Investigations*.³⁵ As a result, most scholarly work on Wittgenstein’s thought either ignores or dismisses the transitional period, focusing instead on the work of the so-called ‘early’ and ‘late’ periods.³⁶ Even the few interpreters who mention it reduce its importance to a mere ‘transitional’ phase between these two more fully developed stages of Wittgenstein’s philosophy. For example, A. C. Grayling writes that “Wittgenstein’s position is essentially the same in these as in the chief works” and “The writings of the transitional period are genuinely transitional, containing elements both of the early and the later views.”³⁷ Nevertheless, the importance of studying Wittgenstein’s thought during this period goes beyond the mere documentation of the transition from the *Tractatus* to the *Investigations*. Wittgenstein’s philosophy in the early thirties is as developed and complete an outlook as that presented in those works.

33. A. C. Grayling, *Wittgenstein* (Oxford: Oxford University Press, 1988), 64.

34. (Canfield 1986, 105)

35. On *The Continuity of Wittgenstein’s Thought*, John Koethe writes: “In the preface to the *Investigations*, Wittgenstein remarks of the *Tractatus* that at one time he had thought that he “should publish those old thoughts and the new ones together: that the latter could be seen in the right light only by contrast with and against the background of my old way of thinking (PI x).” (Ithaca: Cornell, 1996), 4. And Norman Malcolm recounts Wittgenstein confiding that he “thought that in the *Tractatus* he had provided a perfect account of a view that is the *only* alternative to the viewpoint of his later work.” (Malcolm 1984, 69)

36. Dismissing, also, Wittgenstein’s work after the *Philosophical Investigations*.

37. (Grayling 1988, 63)

II. Language, Grammar and Mathematics

Mathematics is no stranger to contemporary linguistics. On the contrary, it has become a very important tool in the scientific study of language. In addition, mathematics is the subject of a considerable strand of linguistic research. For the most part, the linguistic analysis of mathematics is of two different kinds: one which considers mathematics an artificial language,³⁸ and the other which considers mathematics as part of natural language.

Despite the broad differences between mathematics and natural language, many linguists have found enough *language-like attributes*³⁹ in mathematics to justify a linguistic analysis. This sort of linguistic analysis assumes that the syntax and semantics of mathematical statements resembles those of natural language, declarative statements. This justifies applying linguistic tools and theories to the study of mathematics. These linguists see mathematics as an artificial language, independent of natural language. Sometimes, the exaggeration of the separation between mathematical and natural language renders translation impossible.

Mathematicians know this [that mathematics and natural language are too different to translate from one to the other]. Yet they feel ever the compulsion to interpret their mathematics in terms of the every-day language. So proceeding, their harvest is super-paradox.⁴⁰

Less frequently, linguists consider mathematics either part of or derived from natural language. They provide Chomsky-style transformational grammars for the system of numerals⁴¹ or number names.⁴² These linguists consider numerical systems closed regions

³⁸. A. F. Bentley, *Linguistic Analysis of Mathematics*, (Bloomington: Principia Press, 1932)

³⁹. C. F. Hockett, *Language, Mathematics and Linguistics* (La Hague: Mouton, 1967), 6.

⁴⁰. (Bentley 1932, viii)

⁴¹. James R. Hurford, *The Linguistic Theory of Numerals* (Cambridge: Cambridge university Press, 1975)

⁴². Raoul Chapkis & Hugo Brand-Corsitus eds., *Grammars for Number Names* (Dordrecht: Reidel, 1968)

within natural languages.⁴³ In other words, they contemplate numerals only in relation to other numerical expressions. They ignore the occurrence of numerical terms in natural language. For example, the word ‘three’ does not interest them in expressions like ‘there are three computers in this lab’, but only as part of the system of expressions – ‘one’, ‘two’, ‘three’, etc. – for counting. “The ways, that is, in which people in various parts of the world count with words” interests them.⁴⁴ Accordingly, most of these analyses belong to comparative linguistics. Moreover, at the foundation of their studies lies the idea that “the notion of numeration and the concepts of particular numbers are universals, and that the linguistic theory must contain the means for describing how each particular language associates arbitrary phonological sequences (words) with these universal concepts.”⁴⁵

For this sort of linguistic analysis, mathematics is both a tool for linguistic analysis and part of the very language which it analyzes. Wittgenstein, in turn, puts mathematics inside the grammar of language. For him, mathematics is not only a portion of language, but also a component of linguistic grammar. Mathematics is instrument, subject and result of linguistic analysis.

According to Wittgenstein, traditional linguists incorrectly look for a single grammar of language, ignoring the essential multiplicity of linguistic usage. In doing so, grammarians have concentrated on certain uses, while completely ignoring others. Instead of a single grammar, Wittgenstein encourages looking for many grammars corresponding to the many uses of language. One of these grammars is mathematics. For Wittgenstein, the many systems of rules that constitute *mathematics* are nothing but the grammars of diverse linguistic practices. Geometry, for example, is the grammar for describing objects in visual

43. Barron Brainerd, “On the Syntax of Certain Classes of Numerical Expressions” (Chapkis & Brandt 1968, 9)

44. (Hurford 1975, 2)

45. Ibid.

space. Elementary arithmetic, on the other hand, is the grammar of calculating with numbers. Wittgenstein hopes to dissolve most problems on the foundations of mathematics by exposing their grammatical natures. In particular, he hopes to undermine the puzzles behind the applicability and apparent generality, necessity and normative nature of mathematical propositions.⁴⁶

⁴⁶. Within the large bibliography on the intersection between mathematics and linguistics, the idea of mathematical propositions as grammatical is not completely foreign. Consider the following passage from the Preface to Charles F. Hockett's *Language, Mathematics and Linguistics*. "Learning mathematics is like learning any subject, in that one must acquire a new vocabulary. It is like learning a foreign language rather than, say, history, in that one must also acquire alien grammatical habits. And it is like no other subject in that one must also learn how to invent *new* grammatical devices as they are needed."

Chapter 2

On Mathematical Objects and Concepts

I. Introduction

This chapter expounds two essential notions in Wittgenstein's grammatical picture of mathematics: mathematical objects and concepts. Throughout the *Big Typescript*, *Philosophical Grammar* and *Philosophical Remarks*, Wittgenstein develops his ideas on mathematics largely through examples from elementary arithmetic. Following his own presentation, this chapter explores Wittgenstein's ideas through his analysis of numerical expressions. Wittgenstein's ideas respond to Frege's seminal work on the concept 'number'. Wittgenstein criticizes Frege on three important and interrelated matters: (1) the concept-object distinction, (2) the view of equations as identity statements, and (3) the Context Principle.

1. The distinction between concept and object lies at the root of Frege's theory of numerical statements in *The Foundations of Arithmetic* (1884). In his 1892 article, *Concept and Object*, Frege presents the philosophical distinction between concepts and objects as semantical counterpart to the grammatical distinction between proper names and predicates. Wittgenstein bases his major criticism of Frege on this distinction. For Wittgenstein, the distinction between subject and predicate upon which Frege builds his analysis is too crude. It overlooks important logical distinctions among concepts and objects. According to Frege, every statement of true subject-predicate form says that an object – the subject's referent – falls under a concept – the meaning of the predicate. For Wittgenstein, mathematical propositions, even when expressed in subject-predicate statements, do not describe objects conceptually. Instead, they display grammatical relations. Mathematical terms correspond to grammatical categories.

2. For Frege, as well as Frank Ramsey, mathematical equations are identity statements. According to them, arithmetical equations express the reference identity of two numerical expressions. For example, the arithmetical equation ‘ $3 + 4 = 7$ ’ means that numerical terms ‘ $3 + 4$ ’ and ‘ 7 ’ refer to the same object, number seven. For Wittgenstein, mathematical terms are not names. Numerical expressions like ‘ $3 + 4$ ’ and ‘ 7 ’ are not names of numbers; not of the same number, not of any number. Since they do not refer to anything, saying they have the same referent is nonsense. Furthermore, arithmetical expressions like ‘ $3 + 4$ ’, ‘the sum of three and four’ and numerals like ‘ 7 ’ belong to completely different grammatical categories. Terms of the first sort express calculations, while those of the second sort express their final results. Saying that they are both numerical expressions means only that they both belong to the same grammatical system, one which *operates* with numbers, i. e. arithmetic. For Wittgenstein, in general, mathematical equations like ‘ $3 + 4 = 7$ ’ and ‘the derivative of $3x^2$ is $6x$ ’ connect calculations with their final results.

According to Wittgenstein, Frege and Ramsey’s account fails, because treating numerical expressions as names separates numerical identity from syntactic identity. Ramsey and Frege confuse the issue of different numerical expressions *referring* to the same number with them *belonging* to the same number as grammatical category. Different numerical expressions belong to the same number if they obey the same arithmetic rules.

3. Frege’s and Wittgenstein’s philosophical methods are not very dissimilar. Frege’s is a precursor of Wittgenstein’s grammatical method. And Wittgenstein obeyed Frege’s Context Principle, “Never to ask for the meaning of a word in isolation, but only in the context of a proposition.”¹ However, he also thought that Frege did not take his own Context Principle with sufficient seriousness. For Wittgenstein, not asking for the meaning

¹ G. Frege, *The Foundations of Mathematics* (Evanston: Northwestern University Press, 1996), x.

of a word in isolation is not sufficient. Asking for the meaning of a proposition in isolation, outside its larger context of use, is shortsighted too. In the case of mathematical propositions, their contexts are calculi. Every mathematical proposition belongs to some calculus. In every calculus, propositions come in two different sorts. *Calculation propositions* connect calculations with their results. *Specification propositions* designate elements of calculi. In the case of elementary arithmetic, mathematical equations connect calculations with their final results. For example, an arithmetical equation like ‘ $3 + 4 = 7$ ’ says that adding three to four results in seven. A mathematical proposition like ‘7 is a number’, in contrast, specifies an element in the calculus. From this perspective, mathematical propositions like ‘4 is a number’ and equations like ‘ $3 + 4 = 7$ ’ are radically different, since they play essentially different roles in the calculus.

II. Wittgenstein’s Criticism of Frege’s Concept-Object Distinction

A. *Zahlangaben*

Wittgenstein devotes Section XI of the *Philosophical Remarks* to what he terms *Zahlangaben*². A *Zahlangabe* is any expression or diagram equivalent to a statement of the form ‘there are n X s (that are) Y ’, where n is a number, X is a common noun and Y a adjective phrase. This category covers both empirical statements like ‘There is a man on this island’ and mathematical ones like ‘There are 6 permutations of 3 elements’. Furthermore, it also includes arithmetical equations. For example, the equation ‘ $3 + 3 = 6$ ’ is a *Zahlangabe*, because it is equivalent to ‘There are six units in $3 + 3$ ’.

Mann kann auch sagen, der Satz ‘es gibt 6 Permutationen von 3 Elementen’, verhält sich genau so zum Satz ‘es sind 6 Leute im Zimmer’, wie der Satz 3

². Anscombe’s translation of *Zahlangabe* as ‘statement of number’ excludes important cases like diagrams or numerical displays. The expression ‘numerical display’ is closer to the German, because it includes linguistic statements as well as diagrams. This dissertation uses the original German expression.

+ 3 = 6, den man auch in der Form ‘es gibt 6 Einheiten in 3 + 3’ ausprechen könnte. [PR 117 p. 129]

One can also say that the proposition ‘there are 6 permutations of 3 elements’ is related to the proposition ‘There are 6 people in this room’ in precisely the same way as is ‘3 + 3 = 6’, which you could also express in the form ‘There are 6 units in 3 + 3’. [PR §117 p. 139]

Wittgenstein makes it fairly obvious that not all *Zahlangaben* are arithmetical and, furthermore, they are not all mathematical. Wittgenstein offers a definite way to grammatically distinguish mathematical *Zahlangaben* like ‘The interval *AB* is divided into two (3, 4 etc.) equal parts’, from non-mathematical ones like ‘There are four men in this room’.³ For Wittgenstein, the distinction between mathematical and non-mathematical *Zahlangaben* is grammatical. In one important context, they are not exchangeable without grammatical loss. The relevant context is the direct object of the German verb *zu berechnen* [to calculate]. Let *p* be a mathematical *Zahlangabe* and *q* a non-mathematical one. The sentence ‘I calculate whether *p* (is the case/true)’ is correct, but ‘I calculate whether *q* (is the case/true)’ is nonsense. It makes sense to talk of calculating *p*, but not *q*. For example, it makes sense to say ‘I calculate whether there are 5 prime numbers less than 11’ or ‘I calculate whether 3 + 3 = 6’, but not ‘I calculate whether there are four men in this room’ or ‘I calculate whether there are a dozen bowls in my cupboard’. Another way of characterizing this grammatical difference is to let ‘There are *n* *Xs* such that *Y*’ be a mathematical *Zahlangabe*, and ‘There are *m* *Zs* such that *W*’ be a non-mathematical one. It makes sense to say ‘I calculate how many *X* are *Y*’, but not ‘I calculate how many *Z* are *W*’. For example, it makes sense to say ‘I calculate how many permutations are of *AB*’ but not ‘I calculate how many bowls are in my cupboard’.⁴

³. Both examples come from PR Pt. XI §115.

⁴. It might seem that Wittgenstein is wrong in this point. It makes sense to talk about calculating things like the number of bowls in one’s cupboard or whether there are four men in this room. Imagine a case in

Using this grammatical criterion, Wittgenstein divides *Zahlangaben* into two major groups. The first group includes statements like ‘There are 4 men in this room’, ‘There are two circles in this square’ and ‘I have as many spoons as can be put in 1-1 correspondence with a dozen bowls’. The second group contains statements like ‘There are 6 permutations of 3 elements’, ‘ $3+3=6$ ’, ‘There are 6 units in $3+3$ ’, ‘there are 4 prime numbers between 10 and 20’ and ‘A quadratic equation has two roots’. Only the first group expresses genuine [*eigentlich*] propositions. *Genuine* propositions are asserted and negated,⁵ true or false.⁶ Statements in the second group do not describe any possible states of affairs. For Wittgenstein, this means that they are not genuine propositions.⁷ Mathematical *Zahlangaben* are pseudo-propositions.

This distinction is a special case of the more general distinction between genuine and pseudo-propositions. According to Stuart G. Shanker’s *Wittgenstein on the Turning-Point in the Philosophy of Mathematics*, Wittgenstein’s claim that mathematical propositions are grammatical must be understood in the context of Wittgenstein’s comparison of mathematical and genuine propositions (what Shanker calls “empirical propositions”). In particular, it must be perceived in the context of Wittgenstein’s comparison between “the meaning of a mathematical proposition and its method of proof to the meaning of an

which one knows that there are six bowls in each one of the cupboards’ three compartments. In this case, it is correct to say that one can calculate how many bowls are in one cupboard, or that one knows *by calculation* that there are eighteen bowls in the cupboard. However, for Wittgenstein, in these cases, one has not actually calculated whether it is the case that there are eighteen bowls in the cupboard, but whether *it makes sense to say* that there are. The fact that there are such number of bowls in the cupboard is external to the calculus and, in consequence, cannot be settled by calculation alone. The truth of ‘There are eighteen bowls in the cupboard’ still depends on the truth of other non-mathematical *Zahlangaben*: that there are six bowls in each compartment and that there are three compartments in the cupboard. The calculation allows for the transition between these genuine *Zahlangaben*. The role of calculation in these cases is discussed in depth in Chapter 4.

5. PR §163

6. “We could say: a proposition is that to which the truth functions may be applied.” PR §85

7. PR §117

empirical proposition and its method of verification.” Wittgenstein purports to state the similarities and differences between these two sorts of propositions, their meanings and verification methods. He aims at answering two questions: ‘Why are they both propositions?’ and ‘Why are mathematical propositions not empirical?’ Wittgenstein claim that mathematical propositions are grammatical answers both questions.

B. Frege’s Grammatical Enquiry into the Concept of Number

Every good mathematician is at least half a philosopher, and every good philosopher is at least half a mathematician.

Gottlob Frege

Wittgenstein starts his *Zahlangaben* analysis from Frege’s own investigation on *The Foundations of Arithmetic* (1884). In this seminal work, Frege defined the notion ‘cardinal number’ through the primitive notion of a concept’s extension or ‘value-range’. The insight behind Frege’s definition is that a cardinal number statement such as ‘There are n -things’ predicates the number n as a higher-order concept of . Namely, it says that n things fall under . Frege defines the cardinal number of concept (i.e., the number of s) as the concept ‘being a concept equinumerous to ’s extension’. This definition identifies the number of planets as the extension of the concept being a concept ‘equinumerous to the concept of being a planet’. The number of planets is an extension containing all and only those concepts which nine objects exemplify, like the concept ‘being a planet’.⁸ Frege writes,

§46. Um Licht in die Sache zu bringen, wird es gut sein, die Zahl im Zusammenhange eines Urtheils zu betrachten, wo ihre ursprüngliche Anwendungsweise hervortritt. Wenn ich in Ansehung derselben Wahrheit sagen kann: “dies ist eine Baumgruppe” und “dies sind fünf Bäume” oder “hier sind vier Compagnien” und “hier sind 500 Mann,” so ändert sich dabei weder

⁸. Edward N. Zalta, “Gottlob Frege” *Stanford Encyclopaedia of Philosophy* (Palo Alto: Stanford, 1996) [<http://plato.stanford.edu/entries/frege/>]

das Einzelne noch das Ganze, das Aggregat, sondern meine Benennung. Das ist aber nur das Zeichen der Ersetzung eines Begriffes durch einen andern. Damit wird uns als Antwort auf die erste frage des vorigen Paragraphen [von wem durch eine Zahlangabe etwas ausgesagt werde] nahe gelegt, daß die Zahlangabe eine Aussage von einem Begriffe enthalte. Am deutlichsten ist dies veilleicht bei der Zahl 0. Wenn ich sage: “die Venus hat 0 Monde”, dem etwas ausgesagt werden könnte; aber dem Begriffe “Venusmond” wird dadurch eine Eigenschaft beigelegt, nämlich die, nicht unter sich zu befassen. Wenn ich sage: “der Wagen des Kaisers wird von vier Pferden gezogen,” so lege ich die Zahl vier dem Begriffe “Pferd, das den Wzgen des Kaisers zieht,” bei. [p. 59]

§46. It may throw some light on the matter to consider number in the context of a judgement which brings out the way in which it is in origin applied. While looking at one and the same external phenomenon, I can say with equal truth both “It is a copse” and “It is five trees” or both “Here are four companies” and “Here are 500 men.” Now what changes here from one judgement to the other is neither any individual object, nor the whole, the agglomeration of them, but only my terminology. But that is of itself only a sign that one concept has been substituted for another. This suggests as the answer to the first of the questions left open in our last paragraph [when we make a statement of number, what is that of which we assert something?], that a statement of number contains an assertion about a concept. This is perhaps clearer with the number 0. If I say “Venus has 0 moons”, there simply does not exist any moon or agglomeration of moons for anything to be asserted of; but what happens is that a property is assigned to the *concept* “Moon of Venus”, namely that of including nothing under it. If I say “The King’s carriage is drawn by four horses,” then I assign the number four to the concept “horse that draws the King’s carriage.” [p. 59^e]

In 1892, Frege expanded his treatment of the ‘concept’ notion in an article for the *Vierteljahrsschrift für Wissenschaftliche Philosophie*. He used this paper to counteract certain criticisms of the *Foundations*, especially those from Benno Kerry. Frege found that these objections stemmed from a misunderstanding of his ‘concept’ notion. Previously, in the article “Function and Concept” from 1891, he had defined concepts as functions mapping objects to truth values. Later, Frege needed to clarify his distinction between concepts and objects. In his 1892 presentation, he differentiated them through the grammatical distinction between proper names and predicates. For Frege, names refer to objects, while predicates refer to concepts.

The concept (as I understand the word) is predicative [*footnote*: It is, in fact, the referent of a grammatical predicate]. On the other hand, a name of an object, a proper name, is quite incapable of being used as a grammatical predicate.⁹

C. The Grammatical Method

Wittgenstein and Frege shared a strong belief in grammar's philosophical significance. Both found that logical distinctions are ultimately grammatical. For Wittgenstein, inventing distinctions not existing in natural language grammar is idle philosophical speculation. For every philosophical category L , and every expression x , the replacement of 'a' in at least one grammatically acceptable statement by x makes sense if and only if (the meaning of) x belongs to L . In this sense, every significant philosophical distinction and category corresponds to a grammatical one. The grammatical analysis that Wittgenstein endorses in the middle period already appears in many of Frege's arguments. For Frege, grammatical categories are philosophically prior to ontological ones.¹⁰ Two objects or concepts are ontologically different if and only if their names have substantially different grammar. For example, in §29 of the *Foundations of Arithmetic*, Frege argues that the number word 'one' does not stand for a property of objects, because it is not a grammatical predicate.

Wenn "ein Mensch" ähnliche wie "weiser Mensch" aufzufassen wäre, so sollte man denken, daß "Ein" auch als Prädikat gebraucht werden könnte, sodaß man wie "Solon war weise" auch sagen könnte "Solon war Ein" oder "Solon war Einer." . . . Noch deutlicher zeigt sich dies beim Plural. Während man "Solon war weise" zusammenziehen kann in "Solon and Thales waren weise," kann man nicht sagen "Solon und Thales waren Ein." Hiervon wäre die Unmöglichkeit nicht einzusehen, wenn "Ein" sowie "weise" eine Eigenschaft sowohl des Solon als auch des Thales wäre. [Pp. 40, 41]

⁹ G. Frege, "On Concept and Object" in *Translations from the Philosophical Writings of Gottlob Frege*, Geach and Black eds. (Oxford: Basil Blackwell, 1952), 43.

¹⁰ "Frege [da] prioridad a las categorías sintácticas sobre las ontológicas." [Frege gives priority to syntactic categories over ontological ones] Matthias Schirn, "Los Números como Objetos y el Análisis de los Enunciados Numéricos," *Análisis Filosófico* XIV, no. 1 (may 1994): 21-40.

If it were correct to take “one man” in the same way as “wise man”, we should expect to be able to use “one” also as a grammatical predicate, and to be able to say “Solon was one” just as much as “Solon was wise” . . . This is even clearer if we take the plural. Whereas we can combine “Solon was wise” and “Thales was wise” into “Solon and Thales were wise”, we cannot say “Solon and Thales were one.” But it is hard to see why this should be impossible, if “one” was a property both of Solon and of Thales in the same way that “wise” is. [Pp. 40e, 41e]

In this example, the logical category ‘predicate of objects’ corresponds to the context x (‘Solon was x ’). Only expressions the word ‘wise’ substitute for in the statement ‘Solon was wise’ stand for object predicates. ‘One’ is not among them. In consequence, ‘one’ does not stand for a property of objects. In a similar fashion, in §38, Frege argues that one is a unique object, because ‘one’ functions as a proper name. It makes sense to say ‘the number one’, but not ‘a number one’. Furthermore, it does not have a plural form any more than ‘Frederick the Great’, ‘the chemical element gold’ or any other proper name. ‘One’ is a proper noun. In consequence, one is a unique object. For Frege, the category ‘unique objects’ corresponds to the grammatical category ‘proper names’.

D. Wittgenstein’s Criticism of Frege’s Grammatical Analysis of *Zahlangaben*

Not surprisingly, Wittgenstein criticizes the grammatical distinction behind the concept and object characterization in Frege’s analysis of *Zahlangaben*. The core of this criticism appears in the aptly titled Appendix 2 “Concept and Object. Property and Substrate” of the first part of the *Philosophical Grammar*. Wittgenstein questions the philosophical value of Frege’s grammatical analysis of statements in the subject-predicate form.

Begriff und Gegenstand, das ist bei Russell und Frege eigentlich Eigenschaft und Ding; und zwar denke ich hier an einen raumlichen Körper und seine Farbe. Man kann auch sagen: Begriff und Gegenstand, das ist Prädikat und Subjekt. Und das Subjekt-Prädikat Form ist eine Ausdrucksform menschlicher Sprachen. Es ist die Form “ x ist y ” (“ x y ”): “mein Bruder ist groß”, “das Gewitter ist nahe”, “dieser Kreis ist rot”, “August

ist stark”, “2 ist eine Zahl”, “dieses Ding ist ein Stück Kohle.” [PG Pt. I Appendix 2. p. 397]

When Frege and Russell speak of concept and object they really mean property and thing; and here I’m thinking in particular of a spatial body and its colour. Or one can say: concept and object are the same as predicate and subject. The subject-predicate form is one of the forms of expression that occur in human languages. It is the form “x is y” (“x y”): “My brother is tall”, “The storm is nearby”, “This circle is red”, “Augustus is strong”, “2 is a number”, “This thing is a piece of coal.” [PG Pt. I Appendix 2. p. 202]

This interpretation of Frege appears twice in the *Philosophical Remarks*.

Eine Schwierigkeit der Fregeschen Theorie ist die Allgemeinheit der Worte ‘Begriff’ und ‘Gegenstand’. Denn da man Tische und Töne und Schwingungen und Gedanken zählen kann, so ist es schwer, sie alle unter einen Hut zu bringen.

Begriff und Gegenstand, das ist aber Prädikat und Subjekt. Und wir haben gerade gesagt, daß Subjekt-Prädikat nicht *eine* logische Form ist. [PR §93 p. 109]

One difficulty in the Fregean theory is the generality of the words ‘concept’ and object’. For even if you can count tables and tones and vibrations and thoughts, it is difficult to bring them all under one roof.

Concept and object: but that is subject and predicate. And we have just said that there is not just one logical form which is *the* subject-predicate form. [PR §93 p. 119]

Man kann natürlich die Subjekt-Prädikat – oder was dasselbe ist – die Argument-Funktion-Form als eine Norm der Darstellung auffassen, und dann ist es allerdings wichtig und charakteristisch, daß sich in jedem Fall, wenn wir Zahlen anwenden, die Zahl als Eigenschaft eines Prädikats darstellen läßt. Nur müssen wir uns darüber im klaren sein, daß wir es nun nicht mit Gegenständen und Begriffen zu tun haben als den Ergebnissen einer Zerlegung, sondern mit Normen, in die wir den Satz gepreßt haben. Und es hat freilich eine Bedeutung, daß er sich auf diese Norm hat bringen lassen. Aber das In-eine-Norm-Pressen ist das Gegenteil einer Analyse. Wie man, um den natürlichen Wuchs des Apfelbaums zu studieren, nicht den Spalierbaum anschaut, außer, um zu sehen, wie sich *dieser* Baum unter *diesem* Zwang verhält. [PR §115 pp. 125, 127].

You can of course treat the subject-predicate form (or, what comes to the same thing, the argument-function form) as a norm of representation, and then it is admittedly important and characteristic that whenever we use numbers, the number may be represented as the property of a predicate. Only we must be clear about the fact that now we are not dealing with objects and concepts as the results of an analysis, but with moulds into which we have squeezed the proposition. And of course it’s significant that

it can be fitted into this mould. But squeezing something into a mould is the opposite of analysis. If one wants to study the natural growth of an apple tree, one doesn't look at an espalier tree – except to see how *this* tree reacts to *this* pressure. [PR §115. Pp. 136, 137]

Wittgenstein complains that the distinction between subject and predicate at the base of Frege's analysis is undeveloped. "The subject-predicate form serves as a projection of countless different logical forms."¹¹ Frege's analysis is not mistaken, it is limited in scope. It ignores important differences within the 'object' and 'concept' categories. In particular, it fails to distinguish between genuine objects and mathematical ones. It fails to recognize that mathematical concepts are not genuine concepts.

This notation is built up after the analogy of subject-predicate propositions in ordinary language, such as those describing physical objects. . . . And propositions having different grammars, both mathematical and nonmathematical propositions, are dealt with in the same way, e. g., "All men are mortal," "All men in this room have hats," "All rational numbers are comparable in respect to magnitude." [WL *Philosophy for Mathematicians* 1932-33 §1 p.205]

For Wittgenstein, mathematical and non-mathematical *Zahlangaben* involve different concepts. Both sorts of *Zahlangaben* can take the form "There are n X s (such that) Y ". However, the concepts in X and Y are different in each. Concepts such as 'persons', 'spoons', 'this room', 'this square', 'my mother's cupboard' usually occur in non-mathematical *Zahlangaben*. Mathematical *Zahlangaben*, on the contrary, contain terms like 'pure colors', 'units', 'permutations' and numerical expressions like ' $3 + 3$ ', 'two' and 'as many as can be out in 1-1 correspondence with a dozen bowls'. For a *Zahlangabe* to be mathematical, Y must be a calculation concept in the arithmetic of X .¹²

¹¹. PG Pt. I Appendix 2. p. 205.

¹². The following section develops the 'calculation concept' notion, while Chapter 4 explains the expression 'the arithmetic of . . .'. The present chapter analyzes only elementary arithmetic, where X is the concept 'unit'. This section deals exclusively with pure arithmetic, instead of applied arithmetic.

The terms occurring in non-mathematical *Zahlangaben* are genuine names of objects or concepts. The terms in mathematical *Zahlangaben*, on the contrary, do not refer to any genuine concepts. The objects that fall under them are *improper* [*uneigentlich*].

Ja, wir sprechen vom Kreis, seinem Durchmesser, etc., etc. Wie von einem Begriff, dessen Wissenschaften wir beschreiben, gleichgültig, welche Gegenstände unter diesen Begriff fallen. – Dabei ist aber ‘Kreis’ gar kein Prädikat im ursprünglichen Sinn. [PG Pt. I. Appendix 2. p. 404]

We do indeed talk about a circle, its diameter, etc. etc. As if we were describing a concept in complete abstraction from the objects falling under it. – But in that case ‘circle’ is not a predicate in the original sense. [PG Pt. I. Appendix 2. p. 207]

Wittgenstein’s distinction between ‘genuine’ and ‘improper’ concepts and objects springs from his criticism of Frege’s account of *Zahlangaben*. The heart of Wittgenstein’s objection is that Frege’s distinction between object and concept is an insufficient analysis of mathematical *Zahlangaben*. It misses important conceptual differences between mathematical pseudo-propositions and genuine propositions. Frege was blind to the fact that genuine and mathematical concepts’ cardinalities are radically different. Wittgenstein’s criticism clarifies this difference.

E. Mathematical Objects and Concepts

First, Wittgenstein objects to Frege’s analysis of descriptions. Wittgenstein calls ‘description’ any statement of the subject-predicate form. According to Frege, every description says that an object, the referent of the subject, falls under a concept, the referent of the predicate. Wittgenstein considers two different kinds of description: *internal* and *external*.¹³ Internal descriptions ascribe to objects the properties essential for their existence, while external descriptions ascribe accidental properties to them. For Wittgenstein, a

¹³. External descriptions answer the question “which . . . ?”. Internal descriptions answer the question “what sort of . . . ?”. PG Pt. I Appendix 2. p. 398 [P. 204].

property is essential for the existence of an object if its absence “would reduce the existence of the object itself to nothing.”¹⁴ Frege’s analysis holds for external descriptions, but it fails when applied to internal descriptions. In the aforementioned Appendix, Wittgenstein offers the following example:

Was braucht es zu einer Beschreibung, daß – sagen wir – ein Buch an einer bestimmten Stelle ist? Die interne Beschreibung des Buches, d.i. des Begriffes und die Beschreibung seiner Lage, und die wäre durch Angabe der Koordinaten dreier Punkte möglich. Der Satz “Ein solches Buch ist *hier*” würde dann heißen, es hat *diese* drei Bestimmungskordinaten. Die Angabe des Hier darf eben nicht präjudizieren, *was* hier ist. [PG Pt. I, Appendix 2 p. 402]

What is necessary to a description that – say – a book is in certain position? The internal description of the book, i.e. of the concept, and the description of its place which it would be possible to give by giving the co-ordinates of three points. The proposition “Such book is *here*” would mean that it had *these* three coordinates. For the specification of the “here” must not prejudice *what* is here. [PG Pt. I, Appendix 2 pp. 206, 207]

Consider the case in which, pointing at the same object, one makes the following two statements: ‘This book has pages’ and ‘This book is here’. Since having pages is an essential property of any book, the first statement is an internal description. In contrast, the second statement is an example of an external description. It says something about the book. It gives its spatial location. This property is independent of being a book. In the other case, on the contrary, the property of ‘being a book’ already includes ‘having pages’. The internal description does not actually say anything about the object, but about the concept of book under which it falls. The internal description does not describe¹⁵ the book as an object, but the concept ‘book’.

14. PR §94

15. “Property terms in ordinary contexts must stand for qualities that it is sensible to say the substrate *has* or *hasn’t*. It is nonsense to attribute a property to a thing if the thing has been *defined* to have it.” WL *Philosophy for Mathematicians* 1932-33 §3 p. 208.

In this sense, the distinction between internal and external descriptions ultimately depends on the concept under which the described object falls.¹⁶ Instead of an object and a concept, every description involves two concepts. The internal or external nature of the description depends on the relation between these two concepts. If the concept in the subject includes or implies the concept in the predicate, the description is internal. Otherwise, the description is external. The spatial location of a book externally describes it, because the concept of book does not include its location. ‘Having pages’ describes it internally, because the concept ‘book’ includes ‘having pages’. This latter case describes the concept ‘book’, not any particular book. In consequence, Frege’s analysis mistakenly says that every description describes some object. For Wittgenstein, internal descriptions do not describe objects. They state conceptual relations.

As in Wittgenstein’s example of the book, the same object under the same concept can be subject of internal and external description. For Wittgenstein, this means that books are genuine objects. Describing genuine objects both externally and internally is possible. However, other objects may only have internal descriptions. Under some concepts no proper objects may fall. These concepts occur only in internal descriptions. In the aforementioned Appendix, Wittgenstein gives shapes and colors as examples of these concepts. In section XI of the *Philosophical Remarks*, he adds mathematical concepts to this list.

Freilich könnte man so schreiben: Es gibt 3 Kreise, die die Eigenschaft haben rot zu sein. Aber hier tritt der Unterschied zu Tage zwischen den uneigentlichen Gegenständen – Farbflecken im Gesichtsfeld, Tönen, etc. etc. – und den Elementen der Erkenntnis, den eigentlichen Gegenständen. [PR §115 p. 126]

You might of course write it like this: there are 3 circles with the property of being red. But at this point the difference emerges between improper objects

¹⁶. However, only objects under concepts are describable. Even ostensive definitions work only under concepts. PG Pt. I Appendix 2, p. 340 [p.206].

– colour patches in a visual field, etc. etc. – and the elements of knowledge, the genuine objects. [PR §115 p. 136]

For Wittgenstein, mathematical objects are not genuine objects. They have no external properties.¹⁷ All their properties are essential, and essential properties do not describe objects. They describe concepts.¹⁸

All mathematical terms correspond to mathematical concepts. Even in statements of the form subject-predicate, the subject does not refer to any object. For example, the statement ‘4 is a number’ does not ascribe the property of being a number to the object 4. The mathematical statement does not express a proposition of the form $\phi(a)$, because the terms a and ϕ are inseparable.¹⁹ For any mathematical concept, it makes sense to ask whether or not something satisfies it. However, it does not make sense to ask whether or not something that satisfies it *exists*. It makes sense to ask whether or not there are prime numbers between any two given natural numbers, or filters for a determined algebraic structure. However, ‘there are circles’ does not mean that circles exist outside their mathematical role. ‘There are numbers’, ‘circles’, or ‘sets’ only means that those concepts are not extensionally empty. However, the extension of a mathematical concept does not have an existence external to the concept and its intension. In mathematics, ‘there is’ does not equal ‘exists’. Mathematics has no proper *existential* propositions. For Wittgenstein, mathematical propositions of the form ‘ $(x) \cdot \phi x$ ’ do not mean that an object x (such that ϕx) exists. Existential propositions only make sense with genuine concepts. Books exist, but numbers do not. Saying that there are numbers only means that the concept ‘number’ is not empty.

17. Wittgenstein does not seem to take in account those rare cases where mathematical objects have external descriptions, as in ‘3 is Leroy’s favorite number’.

18. PR §94

19. A mathematical proposition $\phi(a)$ is true not because a is ϕ , but because it does not make sense to talk about a not being ϕ . That is why Wittgenstein says that mathematical propositions draw the limits of sense. PR §98.

In summary, Wittgenstein thought that Frege's main flaw was not considering his own Context Principle with sufficient seriousness. If he had, he would have noticed that, just like words, propositions have sense only inside a larger context. He would have noticed that different *Zahlangaben* play different roles in language. Some of them are external descriptions – genuine propositions, while others are internal ones – grammatical rules. Mathematical *Zahlangaben* are the latter kind. The objects and concepts in mathematical propositions are completely different from those in genuine propositions. Mathematical concepts are actually grammatical categories. Mathematical objects are not genuine objects, but roles within a calculus.

III. On Mathematical Equations

A. Frege and Ramsey: Mathematical Equations as Identity Statements

'Every symbol is what it is and not another symbol'.

PR §163 p. 196

Frege did not distinguish between mathematical and non-mathematical *Zahlangaben*. All *Zahlangaben* are arithmetical equations. On §57 of *The Foundations of Arithmetic*, he writes:

Z.B. kann man den Satz "Jupiter hat vier Monde" umsetzen in "die Zahl der Jupitersmonde ist vier.". Hier darf das "ist" nicht als bloße Copula betrachtet werden, wie in dem Satze "der Himmel ist blau". Das zeigt sich darin, daß man sagen kann: "die Zahl der Jupitersmonde ist vier" oder "ist die Zahl 4". Hier hat "ist" denn Sinn von "ist gleich," "ist dasselbe wie". Wir haben also eine Gleichung, die behauptet, daß der Ausdruck "die Zahl der Jupitersmonde" denselben Gegenstand bezeichne wie das Wort "vier". Und die Form der Gleichung ist die Herrschende in der Arithmetik. [P. 69]

For example, the proposition "Jupiter has four moons" can be converted into "the number of Jupiter's moons is four". Here the word "is" should not be taken as a mere copula, as in the proposition "the sky is blue". This is shown by the fact that we can say: "the number of Jupiter's moons is the number four, or 4". Here "is" has the sense of "is identical with" or "is the same as". So what we have is an identity, stating that the expression

“the number of Jupiter’s moons” signifies the same object as the word “four”. And identities are, of all forms of propositions, the most typical of arithmetic. [P. 69^e]

Frege viewed numerical equations – and, in consequence, all *Zahlangaben* – as identity statements. In ‘On Sense and Reference’ [*Über Sinn und Bedeutung*] and *Conceptual Notation* [*Begriffsschrift*], Frege interpreted statements of the form ‘ $a = b$ ’ as identity statements. At the very beginning of ‘On Sense and Reference’, he writes in a footnote,

I use this word [Equality] in the identity sense and I understand ‘ $a = b$ ’ in the sense of ‘ a is the same as b ’ or ‘ a and b agree’.²⁰

Under this point of view, arithmetical equations are similar to identity statements like ‘the morning star is the evening star’. Just like Frege, Ramsey thought of mathematical equations as identity statements. Unlike Frege, Ramsey echoed Wittgenstein’s view of mathematical equality from the *Tractatus* as a relationship between names or signs referring to objects. For Ramsey, they express the referential identity of two nominal expressions. The arithmetical equation ‘ $3 + 4 = 7$ ’, for example, expresses the referential identity between the numerical terms ‘ $3 + 4$ ’ and ‘ 7 ’. In other words, seven is the sum of three and four means that the expressions ‘the sum of three and four’ and ‘seven’ refer to the same number: seven. In his seminal article *The Foundations of Mathematics*, Ramsey writes:

. . . in ‘ $a = b$ ’ either ‘ a ’, ‘ b ’ are names of the same thing, in which case the proposition says nothing, or of different things, in which case it is absurd. In neither case is it the assertion of a fact; it only appears to be a real assertion by confusion with the case when ‘ a ’ or ‘ b ’ is not a name but a description. When ‘ a ’, ‘ b ’ are both names, the only significance which can be placed on ‘ $a = b$ ’ is that it indicates that we use ‘ a ’, ‘ b ’ as names of the same thing or, more generally, as equivalent symbols.²¹

Wittgenstein finds accounts like those of Frege or Ramsey problematic, because by treating numerical expressions as names, they separate the sameness of the number from the

²⁰. G. Frege, “On Sense and Reference” (P. Geach and M. Black 1952), 25-50.

²¹. F. Ramsey, *The Foundations of Mathematics and Other Logical Essays* (New York: Harcourt, Brace & Co., 1931), 180.

sameness of the sign. Wittgenstein, in contrast, does not differentiate between numerical identity and syntactic identity. For him, asking when different signs – different tokens of the same sign-type²² – represent the same number makes no sense. Representing the same number *is* being the same sign.

The question of numerical identity is a question for the identity of sign types. Numerical identity is a grammatical matter.

Ich meine: Die Zahlen sind das, was ich in meiner Sprache durch die Zahlenschemata darstelle.

D.h. ich nehme (sozusagen) als das mir Bekannte die Zahlenschemata der Sprache und sage: Die Zahlen sind das, was diese darstellen [(Spätere Randbemerkung): Statt um eine Definition der Zahl, handelt es sich nur um die Grammatik der Zahlwörter].

Das entspricht dem, was ich seinerzeit meinte, als ich sagte: Die Zahlen treten mit dem Kalkül in die Logik ein. [PR §107]

I mean: numbers are what I represent in my language by number schemata.

That is to say, I take (so to speak) the number schemata of the language as what I know, and say that numbers are what these represent [(Later marginal note): Instead of a question of the definition of number, it's only a question of the grammar of numerals].

This is what I once meant when I said, it is with the calculus [system of calculation] that numbers enter into logic. [PR §107 p. 129]

Accordingly, an answer in terms of mere perception is unsuitable. Seeing directly the number a symbol represents is impossible. Just as it is impossible to *see* that the signs 'a' and 'a' are the same letter, to *see* that '|||||' and '7' are the same number is impossible too.

Wie kann ich wissen, daß ||||| und ||||| *dasselbe* Zeichen sind? Es genügt doch nicht, daß sie *ähnlich* ausschauen. Denn es ist nicht die ungefähre Gleichheit der Gestalt, was die Identität der Zeichen ausmachen darf, sondern gerade eben die Zahlgleichheit. [PR §103. Cf. PG Pt. II §18]

²². Wittgenstein called them 'number schemata'.

How am I to know that ||||| and ||||| are the *same* sign? It isn't enough that they look *alike*. For having approximate similarity in *Gestalt* can't be what is to constitute the identity of the signs, but just being the same in number. [PR §103 p. 125. Cf. PG Pt. II §18 p. 331]

Wittgenstein found that different numerical signs are the same number if and only if they obey the same rules.²³ Determining the numerical identity of different numerical signs requires a grammatical investigation into the rules of the calculus. It must be the result of a 'comparison of the structures' [*Vergleichung der Strukturen* PR §104 p. 117 (p. 126)]. Expressing numerical identity in a mathematical proposition is impossible, because it is not the result of calculation. The problem of numerical identity involves the totality of the calculus. It is not the result of a calculation, but a condition for it. The knowledge that '|||||' and '7' are the same number does not result from arithmetical calculation. A criterion for numerical identity is necessary for doing arithmetic. However, ' $3 + 4 = 7$ ' is the result of an arithmetical calculation and does not express the referential identity of the signs ' $3 + 4$ ' and '7'. Despite both being numerical expressions, they belong to different grammatical categories.

Wittgenstein emphatically rejects the view of mathematical equations as identity statements. Frege bases his interpretation on the view that arithmetical expressions like ' $3 + 4$ ', or 'the sum of three and four' and '7' are names of numbers and that numerals and other, complex numerical expressions are not philosophically different. For Wittgenstein, mathematical terms are not names. Since they do not refer, it does not make sense to talk about them having the same referent.

Second, arithmetical expressions like ' $3+4$ ', or 'the sum of three and four' and numerals like '7' belong to different grammatical categories. According to Wittgenstein's method, grammatical distinctions are prior to logical ones. In this case, calculation expres-

²³. PG Pt. III §15 p. 602 [p. 307]

sions like ‘the product of 3 by 4’ and result expressions like ‘7’ are not exchangeable in the context of the verb to calculate. In other words, it makes sense to say ‘I calculate the product of 3 by 4’, but not ‘I calculate 7’. Mathematical expressions which can substitute for ‘the product of 3 by 4’ in the aforementioned context stand for calculations. They are both numerical expressions, only because they belong to the same grammatical system *operating* with numbers, i. e., arithmetic.

In German grammar, the verb ‘*berechnen*’ [to calculate] is an active or transitive verb.²⁴ It requires a direct object. This means that using ‘*berechnen*’ without a direct object is grammatically incorrect. Determining the calculation object is essential. In a well constructed sentence of German, the direct object usually follows the verb ‘*berechnen*’ in a sentence. Not any sort of expression can serve this function. Answering the question ‘*What is to be calculated?*’ with a verb, adjective or adverb is grammatically incorrect. The answer is always an expression like ‘the successor of four’, ‘the supremum of set E’ or ‘the product of three by four’. These expressions behave like complex nouns. They are either singular or plural. It makes sense to talk about calculating the smallest element in a well ordered set (singular), or about calculating the square roots of a positive number among the reals (plural). And they play the sort of grammatical roles in sentences nouns usually play. Besides the role of direct object, they also play the role of *subject* in certain sentences. For example, it makes sense to say that the supremum of E is less than or equal to any upper bound of E. However, not every mathematical term can play this role. Not any sentence with a numerical expression following the verb ‘to calculate’ makes sense. It makes sense to talk about calculating the product of three by four, but not calculating twelve, for example. It makes sense to talk about calculating the square roots of four, but not about calculating two

²⁴. Unlike the intransitive German verb *rechen*, which also translates to English as ‘to calculate’.

and minus two, even though ‘twelve’, ‘two’, ‘minus two’, just like ‘the square roots of four’ and ‘the product of three by four’ all behave like nouns. In this particular grammatical case, apparently synonymous terms like ‘twelve’ and ‘the product of three by four’ cannot substitute for each other without loss of grammatical correctness.

Mathematical expressions referring to the final results of calculations cannot be direct objects of the verb ‘to calculate’. This latter sort of expressions – sometimes called canonical numerals – behave just like proper names (or lists of them). Talking about *the* number two, instead of *a* number two or *some* number two is grammatically correct. These kinds of mathematical terms are *irreducible*. After arriving at one of them, taking the calculation further is impossible. Instead, the sort of expressions playing the role of direct object of ‘to calculate’ behave more like adjective phrases in natural language. In general, most mathematical terms are either *calculation terms* or *final result terms*. Most philosophers of mathematics conceiving both sort of expressions as names referring to abstract objects overlook this distinction.²⁵ However, it is essential for the analysis of numerical statements.

B. Equations and Calculations

The distinction between calculation expressions and final result expressions sheds some new light on the distinction between obviously tautological equations like ‘ $7 = 7$ ’ or ‘ $3 + 4 = 3 + 4$ ’ and genuine equations like ‘ $3 + 4 = 7$ ’, which have puzzled so many philosophers like Frege. Despite the superficial similarity, in the second sort of equations, the terms on both sides of the ‘=’ are grammatically different. Call them ‘calculation’ and ‘final result terms’ respectively. One expresses a calculation, and the other its final result. In

²⁵. However, not all philosophers have overlooked the distinction. Martin-Löf and Saul Kripke, for example, make a big point of distinguishing terminal and non-terminal mathematical terms.

consequence, the sign '=' in these equations does not work as a *copula*. It connects the calculation with its final result. ' $3 + 4 = 7$ ', for example, is a mathematical proposition saying that seven is the final result of adding three plus four. In it ' $3 + 4$ ' expresses the calculation of adding three plus four, while ' 7 ' is the final result of such calculation.

By contrast, ' $7 = 7$ ' is not a genuine mathematical proposition, because it says nothing about any calculation. Seven is not a calculation. Seven is not the result of calculating seven. Such nonsense results from treating identity statements like ' $7 = 7$ ' as equations on a par with ' $3 + 4 = 7$ '.

Wenn man fragt: Was heißt denn dann aber ' $5 + 7 = 12$ ' – was für ein Sinn oder Zweck bleibt dann für diesen Ausdruck – so ist die Antwort: Diese Gleichung ist eine Zeichenregeln, die angibt, welches Zeichen entsteht, wenn man eine bestimmte Operation (die Addition) auf zwei andere bestimmte Zeichen anwendet. Der Inhalt von $5 + 7 = 12$ ist (wenn einer es nicht wüßte) genau das, was den Kindern Schwierigkeiten macht, wenn sie diesen Satz im Rechenunterricht lernen. [PR §103 p. 116]

If we ask: But what then does ' $5 + 7 = 12$ ' mean – what kind of significance or point is left for this expression – the answer is, this equation is a rule for signs which specifies which sign is the result of applying a particular operation (addition) to two other particular assigns. The content of $5 + 7 = 12$ (supposing someone didn't know it) is precisely what children find difficult when they are learning this proposition in arithmetic lessons. [PR §103 p. 126]

'Die Gleichung ergibt a ' heißt: Wenn ich die Gleichung nach gewissen Regeln transformiere, erhalte ich, also wie die Gleichung $25 \times 25 = 620$ besagt, daß ich 620 erhalte, wenn ich auf 25×25 die Multiplikationsregeln anwende. [PR §150 p. 165]

'The equations yields a ' means: If I transform the equation in accordance with certain rules, I get a , just as the equation $25 \times 25 = 620$ says that I get 620 if I apply the rules of multiplication to 25×25 . [PR §150 p. 175]

Mathematical equations are statements of the form $a = b$ where one of the terms ' a ' or ' b ' is a calculation term and the other is a final result term. In consequence, equations of the form ' $7 = 7$ ' or ' $3 + 4 = 3 + 4$ ' or the false ' $3 = 12$ ', where the expressions on both sides of the '=' sign belong to the same grammatical category, are not calculation statements.

Every mathematical equation of the form $a = b$ or ‘ b is a ’ expresses that b is the final result of calculating a (or vice versa). This holds not only of equations expressed with the help of the ‘=’ sign, but also of equations expressed in prose like ‘two is the positive square root of four’ or ‘seven is the least common denominator of twenty one and fifty six’. The particle ‘is’ does not work as a copula here either. It connects a calculation with its final result. Two is the positive square of four means that two is *the correct final result of calculating* the positive square of four. In general, calculation statements of the form ‘ a is b ’ say that a is the correct result of calculating b . Despite their surface grammar, these are not propositions of the form $F(x)$, where x is an object, and F is a property. The number seven is not an object, and being the least common denominator of twenty-one and fifty-six is not one of its properties. Wittgenstein addresses this in §102 of the *Philosophical Remarks*, where he says that using ‘=’ can make numerical assertions appear to refer to genuine concepts, when they do not.

IV. An Extension of Frege’s Context Principle

A. The Context Principle

The Context Principle is a fundamental principle [*Grundsätze*] of Frege’s philosophical enquiry. In the introduction to his *Foundations of Arithmetic*, he formulates this principle as “never to ask for the meaning of a word in isolation, but only in the context of a proposition” [*nach der Bedeutung der Wörter muss in Satzzusammenhänge, nicht in ihrer Vereinzelung gefragt werden*].

But we ought always to keep before our eyes a complete proposition. Only in a proposition have the words really a meaning. It may be that mental pictures float above us all the while, but these need not correspond to the logical elements in the judgement. It is enough if the proposition taken as a whole has a sense; it is this that confers on its parts also their content.²⁶

²⁶. (Frege 1950, 71)

Wittgenstein thought that the main flaw of Frege's analysis was not considering his own Context Principle seriously enough. Wittgenstein agrees with Frege that words have meaning only in the context of a proposition. However, Wittgenstein also recommended asking for the meaning of a proposition not in isolation but in its larger context of use. For mathematical propositions, this context is their use in calculation. Wittgenstein's extension of Frege's Context Principle commands asking for the meaning of mathematical propositions only in the context of their calculi.

B. Specification and Calculation Propositions and Concepts

According to Frege's analysis, both mathematical *Zahlangaben* like '4 is a number' and equations like ' $3 + 4 = 7$ ' are propositions of the subject-predicate form $F(a)$, where 'F' is a predicate – referring to a concept – and 'a' is a name – whose referent is an object. Furthermore, since all *Zahlangaben* are arithmetical equations, the concept involved in a *Zahlangabe* is always of the form ' $\dots = b$ ' where 'b' is also the name of a number. For Wittgenstein, Frege's distinction overlooks the real difference between these two sorts of mathematical propositions. This difference stems from the different roles they play in their calculi. Both sorts of propositions are rules of mathematical calculi. But they are rules of different sorts. Every calculus has two different sorts of propositions. Propositions of one sort connect calculations with their results. Call these '*calculation propositions*'. Propositions of the second sort specify the calculus' elements. Call propositions of this sort '*specification propositions*'. In elementary arithmetic, the calculation propositions are the equations. Mathematical equations connect calculations with their final results. For example, an arithmetical equation like ' $3 + 4 = 7$ ' says that adding three plus four results in eight. A mathematical proposition like '7 is a number' does not. It says that '7' belongs to the category of number. 'Being a number' is not a calculation, but a calculus category. A

category's elements share the same role in the calculus. Saying that 7 is a number specifies the role of '7' in the calculus. Frege overlooks this difference as well, while Wittgenstein assigns it a central role in his philosophy of mathematics.²⁷

A previous section showed how most arithmetical terms in a calculus are either calculation or result terms. However, not all mathematical terms fit into these categories. Consider the term 'number'. Even though basic in arithmetic, 'number' is neither a result nor a calculation term. It does not occur in either side of the '=' sign in arithmetical equations.²⁸ Other examples of arithmetical terms of this sort are 'addition', 'unit', 'equation', etc. They also describe calculation and result terms internally, but they are not calculation or result terms themselves. They occur in internal descriptions not corresponding to any calculation whatsoever. Call these descriptions '*specification propositions*'.

The distinction between calculation propositions and specification propositions corresponds to a distinction at the level of mathematical concepts. The concepts that occur in specification propositions differ from those in calculation propositions. On a superficial level, statements of the subject + predicate form express propositions of both sorts. However, the concepts that occur as predicates in the specification propositions do not occur as subjects or predicates in calculation statements.

In the case of elementary arithmetic, 'number', 'addition', 'equation', etc. are specification concepts. In consequence, '4 is a number' and '3 + 4 is an addition' are specification propositions. They differ from calculation propositions in that they do not describe the result of any calculation in the calculus they specify. They can play the role of calculation propositions in other calculi, but not in the calculus they specify. For example,

²⁷. This distinction is essential for discussing Wittgenstein's account of mathematical modality.

²⁸. It certainly occurs in arithmetical statements, but its role is singular. It expresses equations in prose. The statement '7 is the sum of *numbers* 3 and 4' expresses the equation '3 + 4 = 7'. However, in these cases, it still does not express a calculation or its result.

constructing a calculus where propositions like ‘4 is a number’ or ‘ $3 + 4 = 7$ is an equation’ are calculation propositions is possible. Wittgenstein found that much of the logicians’ work on the foundations of arithmetic is of this sort. What Frege achieved by giving a formal definition of number was a new calculus in which propositions like ‘4 is a number’ are calculation propositions. He has constructed a calculus where the concept of number is not a specification one, but a calculation one. Hence, it makes sense to say in Frege’s framework ‘I calculate whether 4 is a number’.

In summary, mathematical propositions may be either calculation or specification propositions. Calculation propositions connect calculations with their results. Specification propositions distribute the elements of the calculus into categories.

V. Beyond Arithmetic, Proofs as Calculations

Following Wittgenstein’s own presentation, this chapter has focused mainly on arithmetic. However, Wittgenstein’s notion of calculation covers other mathematical processes, like counting, drawing geometrical figures, and proving theorems within a formal system. Hence, his theory of calculation constitutes a general philosophy of mathematics. Consider, for example, the important case of proving a theorem within a formal system. According to Wittgenstein, proving a theorem within a system is a calculation just like adding or multiplying. The only difference is that its final result is a mathematical proposition, instead of a number.

Compare the grammatical analysis of calculation to proving theorems. The transitive uses of the verbs ‘to calculate’ and ‘to prove that’ in mathematics differ in the expressions they accept as direct complements. Their direct complements are grammatically different. In the case of ‘to calculate’, the direct object is a nominal phrase. However, this is not the case for the verb ‘to prove that’. Unlike the verb ‘to calculate’, the verb ‘to prove that’ does not accept calculation terms, like ‘ $3+4$ ’ or ‘the supremum of set E’, as their direct comple-

ments. It does not accept result terms either. Its direct complement is a mathematical statement. The only exception is descriptive nominal phrases which refer to mathematical statements. In mathematics, a mathematical statement often follows the expression ‘to prove that’. In other words, the direct object is a mathematical statement.²⁹ The proof’s object is a mathematical statement.³⁰ Genuine propositions are unprovable. It makes sense to talk about proving that $3 + 4 = 7$ or that every equation of second degree has two roots. However, it does not make sense to ask for the proof that there are twenty-four pages in this chapter or fifty states in the Union. The latter propositions are genuine ones, not mathematical. Only mathematical propositions are provable.³¹ In consequence, mathematical propositions are themselves result terms corresponding to the calculation of proof. Wittgenstein’s analysis of calculation applies to them as well.

At first sight, calculating in arithmetic and proving theorems in a formal system seem to be very different, because most mathematical calculations are operations. They have one and only one correct final result. However, the concept ‘being a theorem of a system’ usually covers a multiplicity of propositions. While 25×25 has only one product, there are many theorems of PA. In consequence, proving a theorem seems to be not a calculation. Calculations are univocal processes. Nevertheless, this does not present a problem for Wittgenstein’s theory. For him, ‘being a theorem’ is not a calculation concept. It is a

²⁹. Even in cases where the direct object is not a formula, the non-symbolic statement also expresses a mathematical proposition. Wittgenstein scorns mathematical prose, because expressions with non-formal meaning are confusing. For a statement to be mathematical, the meaning of all its parts must be purely mathematical. Every word must make sense only inside the system of calculation. Mathematical propositions cannot say anything else, and to phrase them in a way that may suggest otherwise is philosophically misleading.

³⁰. Mathematical propositions of the form ‘ $\vdash p$ ’ or ‘ p is a theorem’ are mathematical, because they express that ‘ p ’ is the result of the calculation of proving it as a theorem.

³¹. Cf. Chapter 4 for a detailed account of the role mathematics plays in proofs of non-mathematical propositions.

specification concept, like ‘product’ or ‘root’. Mathematical expressions of the form ‘ $|p$ ’ or ‘ p is a theorem’, are not calculation propositions, but specification ones. Just as many numbers are roots or products, so many propositions are theorems. Furthermore, just as saying that one number is a product means that it is the final result of some multiplication, a proposition being a theorem means that it results from a proof. In this sense, the proof is the calculation. Consequently, the calculation propositions of this calculus are the units of theorems and their proofs.

Man könnte auch so sagen: Der völlig analysierte mathematische Satz ist sein eigener Beweis.

Oder auch so: der mathematische Satz ist nur die unmittelbar sichtbare Oberfläche des ganzen Beweiskörpers, den sie vorne begrenzt.

Der mathematische Satz ist – Im Gegensatz zu einem eigentlichen Satze – *wesentlich* das letzte Glied einer Demonstration, die ihn als richtig oder unrichtig sichtbar macht [PR §162 p. 182]

We might also put it like this: the completely analysed mathematical proposition is its own proof.

Or like this: a mathematical proposition is only the immediately visible surface of a whole body of proof and this surface is the boundary facing us.

A mathematical proposition – Unlike a genuine proposition – is *essentially* the last link in a demonstration that renders it visibly right or wrong. [PR §162 p. 192] ³²

Just as every calculation has a unique result, mathematical propositions have unique proofs.

Mathematical theorems with different proofs are different propositions. For Wittgenstein, calculations and proofs are rule-governed activities. They have one single and unique result.

A calculation failing to arrive at any result is not a calculation, but an idle, aimless wander.

Talking about calculations with more than one result is nonsense too. The obvious case of

³². “A mathematical proposition is related to its proof as the outer surface of a body is to the body itself. We might talk of the body of proof belonging to the proposition. Only on the assumption that there’s a body behind the surface, has the proposition any significance for us. [PR §162 p. 192] Here again, one can only say: look at the proof, and you will see *what* is proved here, what gets called “the proposition proved.” [PG Pt. II §13 p. 301] [I have] likened the conclusion of a mathematical proof to the end surface of a cylinder. The proved proposition is the end surface of the proof, *a part of it*. . . . But in mathematics the proof is not a symptom, because the proved proposition is part of the proof. [WL *Philosophy for Mathematicians* 1932-33 §10 p. 221]

equations with more than one root is not a counter example, because it actually hides a multiplicity of mathematical propositions. The process of arriving at each equation's root is different. Each one is a different calculation. Furthermore, the process of arriving at the knowledge that they are both roots of the equation is yet another calculation. In either case, the connection between process and result is so intimate that they cannot be understood in isolation. They provide meaning for each other.

VI. Conclusion

This chapter expounds on the essential notions in Wittgenstein's claim that mathematical propositions are ultimately grammatical. Throughout the *Big Typescript*, *Philosophical Grammar* and *Philosophical Remarks*, Wittgenstein develops his ideas on mathematics largely through examples from elementary arithmetic. Following his own presentation, this chapter begins exploring Wittgenstein's ideas through his analysis of numerical expressions.

Wittgenstein developed his ideas in response to Frege's seminal work on the concept of number. Wittgenstein shares two substantial, methodological principles with Frege: the Grammatical Principle and the Context Principle. For Frege, as well as Wittgenstein, grammatical distinctions have strong philosophical significance. Both believe that every significant philosophical distinction is already present in the grammar of language. For them, the meaning of two expressions *is* logically different if and only if their replacement affects the grammar of the expression in which they occur.

Frege and Wittgenstein also share a strong faith in the philosophical importance of context. For Frege, words have meaning only in the context of propositions. Understanding the meaning of a word requires analyzing its role in complete sentences. Wittgenstein extends Frege's Context Principle to cover sentences as well as words. For him, under-

standing the meaning of a sentence requires an analysis of its role in a larger system of propositions (or in other sentences), too. In particular, understanding the meaning of numerical statements [*Zahlangaben*] requires an analysis of their roles in the contexts of their use. In the case of mathematical *Zahlangaben*, it requires understanding their roles in calculation. In the case of non-mathematical ones, it requires an analysis of their role in the application of mathematics. Both analyses are essential for the full understanding of numbers in mathematics. If mathematical propositions are grammatical, this would manifest itself in their roles in both calculation and application. The next two chapters follow Wittgenstein's investigation of mathematical calculation and application. The results from both investigations paint a full picture of the grammatical role of mathematical *Zahlangaben* in particular, and mathematical propositions in general.

Chapter 3

The Grammar of Calculation

I. Introduction

Die Mathematik besteht ganz aus
Rechnung.

PG §40 p.924

Mathematics consists entirely of
calculation.

PG §40 p. 468

The previous chapter introduced Wittgenstein's thesis that mathematical propositions connect calculations with their final results. Understanding why this also means that the mathematical proposition is a grammatical rule of its calculus requires conceiving of calculation as a rule-governed linguistic practice and of calculi as grammatical systems. This chapter addresses calculation in Wittgenstein's grammatical account of mathematics. Wittgenstein claims that the final result of calculation is correct only if a correct performance of the calculation produces it. Every mathematical calculus is a linguistic system with its own grammar. This grammar provides the criteria for correct and incorrect results. Wittgenstein bases his vision of mathematics as grammar on the claim that all mathematical problems – problems solvable by counting, calculating, drawing a geometrical figure, or proving a theorem in a formal system — are ultimately problems of deciding whether or not an expression fits in a mathematical category. For the most part, this argument appears in Part II Section V of *Philosophical Grammar*, and Section XIII of the second part of the *Philosophical Remarks*. Wittgenstein contrasts his account of calculation with the more traditional Platonistic account, where calculations are *explorations* into some yet unmapped, mathematical territory. Wittgenstein views calculations more as rule-governed *searches* over well-defined grammatical spaces.

II. Mathematics as Calculation

A. “There’s a *Fascination* here:” Mathematical Problems and Problems of Mathematical Investigation

Das Unglück ist, daß unsere Sprache so grundverschiedene Dinge mit jedem der Worte ‘Frage’, ‘Problem’, ‘Untersuchung’, ‘Entdeckung’ bezeichnet.

PG §22 p.706

The sad thing is that our language uses each of the words ‘question’, ‘problem’, ‘investigation’, ‘discovery’ to refer to such fundamentally different things.

PG §22 p. 359

At §22, the very beginning of Section V of the *Philosophical Grammar*’s second part, Wittgenstein distinguishes between tasks of calculation [*Aufgaben der Rechnung*] (also called ‘mathematical problems’ [*mathematische Probleme*] or ‘problems in mathematics’ [*Probleme in der Mathematik*]) – and problems of mathematical investigation [*Problem der mathematischen Forschung*] (also called ‘problems of mathematics’ [*Probleme der Mathematik*]), such as Fermat’s Last Theorem or Riemann’s Hypothesis. There, he writes:

‘Wenn auf die Lösung – etwa – des Fermat’schen Problems Preise ausgesetzt sind, so könnte man mir vorhalten: Wie kannst Du sagen, daß es dieses Problem wohl nicht geben. Ich müßte sagen: Gewiß, nur mißverstehen die, die darüber reden, die Grammatik des wortes “mathematisches Problem” und des Wortes “Lösung”. Der Preis is eigentlich auf die Lösung einer naturwissenschaftlichen Aufgabe gesetzt; auf das *Äußere* der Lösung (darum spricht man z. B. auch von einer Riemann’schen *Hypothese*). Die Bedingungen der Aufgabe sind äußerliche; und wenn die Aufgabe gelöst ist, so entspricht, was geschehen ist, der Stellung der Aufgabe, wie die Lösung einer physicalischen Aufgabe dieser Aufgabe. [p. 712]

Suppose prizes are offered for the solution – say – of Fermat’s problem. Someone might object to me: How can you say that this problem doesn’t exist? If prizes are offered for the solution, then surely the problem must exist. I would have to say: Certainly, but the people who talk about it don’t understand the grammar of the expression “mathematical problem” or the word “solution”. The prize is really offered for the solution of a scientific problem; for the *outside* of the solution (hence also for instance we talk about Riemann’s *Hypothese*). The conditions of the problem are external conditions; and when the problem is solved, what happens corresponds to the setting of the problem in the way in which solutions correspond to problems in physics [p. 362]

Under the vocabulary developed in the first chapter, a mathematical problem always asks if a calculation proposition is correct. Every mathematical problem tests the correctness of a calculation proposition. A problem is mathematical only if it can be formulated as a question ‘Is p so?’, where p is a calculation proposition.

The mathematician’s activity is carried on in a particular sphere. A question is part of a calculus. What does it prompt you to do? [WL *Philosophy for Mathematicians* 1932-33 §10 p. 222]

Nur dort kann man in der Mathematik fragen (oder vermuten), wo die Antwort lautet: ‘Ich muß es ausrechnen’. [PR §151 p. 165]

We may only put a question in mathematics (or make a conjecture), where the answer runs: ‘I must work it out’. [PR §151 p. 175]

A mathematical proposition is true in a completely different sense than other kinds of propositions, whether they are specification propositions or non-mathematical ones, are true. In the case of a calculation proposition p , p is true means that p is correct, according to the calculus. Since a calculation proposition connects a calculation to its result, a calculation proposition is correct if and only if it connects a calculation with its correct result. The calculation ultimately bears the solution of a mathematical problem.

The boundaries between mathematical problems and problems of mathematical investigation are sometimes difficult to see. In §23, Wittgenstein examines whether or not the equation ‘ $x^2 + ax + b = 0$ ’ has a solution over the real numbers. Whether or not any real number satisfies the equation seems to be not a matter of mere calculation, but an ontological question about the existence of an abstract object with a certain property. However, it is, in fact, a mathematical problem equivalent to the question ‘(n) $n^2 + an + b = 0$?’ In consequence, calculating its solution is possible. The prose of the original question obscured its mathematical nature. In mathematical prose, the expressions ‘there is’ and ‘exists’ cor-

respond to the formal symbol ‘ \exists ’.¹ Outside mathematics, however, they mean something completely different.

In Wirklichkeit ist Existenz das, was man mit *dem* beweist, was man “Existenzbeweis” nennt. . . Wir haben keinen Begriff der Existenz unabhängig von unserm Begriff des Existenzbeweises. [PG Pt. II §24 p. 736]

Really, existence is what is proved by what we call “existence proof”. . . We have no concept of existence independent of our concept of existence proof. [PG Pt. II §24 p. 374]

Problems of mathematical investigation are not problems of mathematical calculation. They can be formulated as questions ‘Is p so?’. However, here, p is not a calculation proposition. Instead, p is a genuine *hypothesis*. Problems of mathematical investigation seem to be a call for calculating what cannot be calculated. Mathematical problems are tasks of calculation that demonstrate the correctness of calculation propositions. By contrast, problems of mathematical investigation are pseudo-mathematical problems. Problems of mathematical investigation are of two kinds: (1) genuine problems disguised as mathematical problems or (2) ungrammatical pseudo-problems. In the first case, they pose external questions about the calculi. Since “calculus is not a mathematical concept” [*Kalkül ist kein mathematischer Begriff* PG Pt. II §12 p. 580 (p. 296)], calculations cannot say anything about their own calculi. No *metacalculi* exist. For example, the question if a given calculus has any external application is an empirical question disguised as a mathematical one. The calculus itself cannot answer this question. “That will show itself soon” [*das wird sich dann schon zeigen* PG Pt. II §15 p. 600 (p. 306)] is the only possible answer. Applying the calculus is the only way to demonstrate its applicability. So-called mathematical conjectures [*Vermutungen*] of this kind are actually physical hypotheses, not mathematical ones.

¹. PG Pt. II §24

Other problems of mathematical investigation are not-yet-mathematical problems. These questions are strings in mathematical language whose surface grammar is similar to that of calculation propositions. However, they cannot be calculated yet because they do not fit in any existent calculus. They do not amount to actual mathematical problems, because their calculation does not belong to any current calculus. Fermat's Last Theorem is a problem of this sort. The symbols in the expression $x^n + y^n = z^n$? certainly belong to a mathematical calculus. However, their combination does not, yet.

Ich behaupte ja nicht: wenn jemand sich mit dem Fermatschen Problem beschäftigt, so ist das falsch oder unberechtigt. Durchaus nicht! Wenn ich z. B. eine Methode habe, um nach ganzen Zahlen zu suchen, welche die Gleichung $x^2 + y^2 = z^2$ erfüllen, so kann mich die Formel $x^n + y^n = z^n$ anregen. Ich kann mich von einer Formel anregen lassen. Ich werde also sagen: hier liegt eine *Anregung* vor, aber keine *Frage*. Die mathematische 'Probleme' sind immer solche Anregungen. Diese Anregungen sind nicht etwa eine Vorbereitung auf einen Kalkül. [PR Appendix II, p. 321]

I'm certainly not saying: if anyone concerns himself with Fermat's Last Theorem, that's wrong or illegitimate. Not at all! If, for instance, I have a method for looking for whole numbers satisfying the equation $x^2 + y^2 = z^2$, the formula $x^n + y^n = z^n$ can disturb me. I can allow myself to be disturbed by a formula. And so I shall say: there's a *fascination* here but not a *question*. Mathematical 'problems' always stir us up like this. This kind of fascination is not really the preparation of a calculus. [PR Appendix II, p. 334]

Der Fermatische Satz hat also keinen *Sinn*, solange ich nach der Auflösung der Gleichung durch Kardinalzahlen nicht *suchen* kann. [PR §150 p. 165]

Thus Fermat's Theorem makes no *sense* until I can *search* for a solution to the equation in the cardinal numbers. [PR §150 p. 175]

Solving this kind of problem requires the construction of a new calculus where that combination of signs makes sense. It is necessary to construct a calculus where a calculation answers the question in the old problem. Only then, the question becomes meaningful and a proper mathematical problem. However, then, calculation does not solve the original pseudo-problem of mathematics. It solves the new mathematical one.

In the end, Wittgenstein finds that the only thing these two different sorts of problems have in common is that they have a solution.² However, only one of them has a method.

B. Method and Solution

The mathematical processes involved in the solution of mathematical problems are rule-governed activities. Calculating, proving theorems within a formal system, *et cetera* are all activities governed by rules. These rules provide the criteria determining whether their results –and, hence, the solution to the mathematical problem – are correct or incorrect. For example, the question ‘ $25 \times 25 = 625?$ ’ is mathematical only if calculation can solve it. If a general method of solution exists, the question is mathematical. An addition is a mathematical problem if its calculation follows a set of rules. The correct answer to the question ‘ $25 \times 25 = 625?$ ’ results from following the rules throughout the calculation. In §25, Wittgenstein writes:

“Du sagst ‘Wo eine Frage ist, da ist auch ein Weg zu ihren Beantwortung’, Aber in der Mathematik gibt es doch Fragen, zu deren Beantwortung wir keinen Weg sehen.” – Ganz richtig, und daraus folgt nur, daß wir in diesem Fall das Wort ‘Frage’ in anderem Sinn gebrauchen, als im oberen Fall. Und ich hätte vielleicht sagen sollen “es sind hier zwei verschiedene Formen und nur für die erste möchte ich das Wort ‘Frage’ gebrauchen.” Aber dieses Letztere ist nebensächlich. Wichtig ist, daß wir es hier mit zwei verschiedenen Formen zu tun haben. (Und daß Du Dich in der Grammatik des Wortes ‘Art’ nicht auskennst, wenn Du nun sagen willst, es seien eben nur zwei verschiedenen *Arten* von Fragen.) [PG §25 p. 748]

“You say ‘Where there is a question, there is also a way to answer it’, but in mathematics there are questions that we do not see any way to answer.” Quite right, and all that follows from that is that in this case we are not using the word ‘question’ in the same sense as above. And perhaps I should have said “Here there are two different forms and I want to use the word ‘question’ only for the first”. But this latter point is a side-issue. What is important is that we are here concerned with two different forms. (And if

². PR §151

you want to say they are just two different *kinds* of question you do not know your way about the grammar of the word “kind.”) [PR 25 p. 380]

If a question is asked for which there does not exist a method of solution, does the question have meaning? I have said “No.” [WL *Philosophy for Mathematicians* 1932-33 §10 p. 221]

At the beginning of Section 23 of *Philosophical Grammar*, Wittgenstein states that the truth of a calculation proposition is not discovered –or invented for that matter, but *checked*. For Wittgenstein, systems of propositions define methods of solution. The system of propositions contains those propositions which are correct according to it. In consequence, a proposition expresses a correct calculation if it is one of the rules defining the calculation. Checking the truth of a calculation proposition, is looking for it among the propositions defining the general method of solution.

C. Calculations are Transitions between Expressions According to a Rule

Ich will also sagen: das Arithmetische ist nicht der Anlaß, 5 und 7 zusammenzugeben, sondern der Vorgang und was dabei herauskommt.	I want to thus say: arithmetic is not the cause to combine 5 and 7 but the process and its outcome.
	PR §104

PR §104

Calculations are not events. They are rule-governed practices. This distinction lies at the heart of Wittgenstein’s discussion between ‘following’ and ‘inferring’ at the beginning of the *Remarks on the Foundations of Mathematics*. For Wittgenstein, inferring is an activity. It is something people do. People infer some propositions from others. Whether or not these propositions follow from each other is a very different matter. That a proposition follows from another does not require anyone actually inferring it. Logic is not natural history. That a proposition *A* follows from *B* does not mean that we — whoever *we* are — *usually* infer *A* from *B*. It relies on the logical rules of inference. In the case of logic or mathematics, ‘to follow’ is a grammatical relationship between mathematical propositions.

Das kann auf dem Papier, mündlich, oder 'im Kopf' vor sich gehen. - Der Schluß kann aber auch so gezogen werden, daß der eine Satz, ohne Überleitung, nach dem andern ausgesprochen wird; oder die Überleitung besteht nur darin, daß wir "Also", oder "Daraus folgt" sagen, oder dergleiche [RFM §6 p. 39].

This [inferring] may go on paper, orally or 'in the head'. —The conclusion may however also be drawn in such a way that the one proposition is uttered after the other, without any such process; or the process may consist merely in our saying "Therefore" or "It follows from this", or something of the kind [RFM §6 p.5].

On the other hand, 'Following' has nothing to do with assertions – i.e. utterances [*Behauptung*] – which are events themselves, but with propositions [*Satz*]. The 'following' of one mathematical proposition to another is a transition. However, it is not a process or event. Following does not happen in time.

Wenn wir sagen: "dieser Satz folgt aus jenem", so ist hier "folgen" wieder *unzeitlich* gebraucht. (Und das zeigt, daß dieser Satz nicht das Resultat eines Experiments ausspricht.) [RFM §103 p.30^e]

When we say: "This proposition follows from that one" here again "to follow" is being used *non-temporally*. (And this shows that the propositions does not express the result of an experiment.) [RFM §103 p.30]

Unlike drawing conclusions in everyday life, logical inference results from calculation. Wittgenstein defines inference as "the derivation of one sentence [*Satz-zeichen*] from another according to a rule."³ Derivation is a logical calculation whose result is a logical inference. The forming [*Bilden*] of the derivation is "the comparison of both sentential expressions [*Satz-zeichen*] with some paradigm or another which represents the schema of the transition." Consider the derivation of the sentence '(PvQ)' from '(PvQ)&R', according to the rule of conjunction elimination. The derivation's first step is to take a scheme *representing* the rule:

³. RFM §6

$$\frac{A \& B}{A}$$

Second, compare the two sentences to those in the paradigm.⁴ Wittgenstein recommends arranging them in the form of the scheme:

$$\frac{(P \vee Q) \& R}{(P \vee Q)}$$

Comparing the signs in the calculation with those in the scheme allows for the formation of the derivation. Recognizing the adequate correspondence justifies the correctness of the derivation. More complex derivations require the reiteration of these simple steps.

All calculations follow this simple process. Every calculation is a transition from one expression to another according to a rule. The ‘forming’ of every step in the calculation involves the comparison of expressions with some paradigm representing the schema of the transition.⁵

III. Mathematical Correctness

Man kann Mathematik nicht schreiben, sondern nur machen.	You can't write mathematics, you can only do it.	PR §157
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PR §157

A calculation proposition and other kinds of propositions – specification propositions and non-mathematical ones – are true in completely different senses. In the case of a calculation proposition p , that p is true means that p is the correct result of a calculation performed in accordance with the rules of the calculus. The calculation always bears the solution of a mathematical problem. In consequence, it is more proper to talk about mathematical

⁴ RFM §31

⁵ Sometimes the paradigm may not be physically present, but “in the head.” For example, having the tables of multiplication at hand is not necessary for multiplying 3 x 4.

correctness than mathematical *truth*. The contrary of a true mathematical proposition is not a false one, but nonsense resulting perhaps from an incorrect calculation.

Was ist das Gegenteil des Beweisenen? – Dazu muß man auf den Beweis schauen. Man kann sagen: das Gegenteil des beweisenen Satzen ist das, was statt seiner durch einen bestimmten Rechnungsfehler im Beweis bewiesen worden wäre. [PG Pt. II §24 p. 732]

What is the contradictory of what is proved? – For that you must look at the proof. We can say that the contradictory of a proved proposition is what would have been proved instead were a particular miscalculation made in the proof. [PG Pt. II §24 p. 372]

Calculations cannot be false, but they can be incorrect.

A. Wittgenstein’s Anti-Platonism

1. Platonist Mathematical Explorations

As part of Wittgenstein’s criticism of Frege’s philosophy of mathematics, Wittgenstein takes an anti-platonist attitude regarding calculations. Frege’s position is usually characterized as a kind of platonism. Frege wrote that numbers are “self-subsistent”⁶ objects with definite “properties that can be specified.”⁷ He also wrote that “the mathematician cannot create things at will, any more than the geographer can; he too can only discover what is there and give it a name.”⁸ Of course, Frege did not originate the idea that mathematics consists of a body of discoveries about an independent reality made up of objects like numbers, sets, shapes and so forth. Furthermore, platonism did not receive its name until 1934, when Paul Bernays baptized it in his lecture ‘On Platonism in Mathematics.’ He characterized ‘Platonism’ as the view of mathematical objects “as cut off from all links with

6. G. Frege, *The Foundations of Arithmetic* (Evanston: Northwestern University Press, 1950), §55

7. Ibid. §8

8. Ibid. §96

the reflecting subject.” He continued, “Since this tendency asserted itself especially in the philosophy of Plato, allow me to call it ‘Platonism’.”

Many mathematicians enthusiastically adhere to the ideals of Platonism. From Frege’s contemporaries like Emile Borel and Charles Hermite, to more recent thinkers like Kurt Gödel and Roger Penrose, several eminent mathematicians have embraced some form of Platonism. Nevertheless, Platonism is not a unitary doctrine or a definite philosophical theory. It is a cluster of metaphors that together make up a heuristic picture of mathematics. The *Macmillan Encyclopedia of Philosophy*, edited by Paul Edwards⁹, offers the following definition of *platonism*,

By platonism is understood the realist view, akin to that of Plato himself, that abstract entities exist in their own right, independently of human thinking. According to this view number theory is to be regarded as the description of a realm of objective, self-subsistent mathematical objects that are timeless, non-spatial, and non-mental. **Platonism conceives it to be the task of the mathematician to explore this and other realms of being.** Among modern philosophers of mathematics Frege is a pre-eminent representative of platonism, distinguished by his penetrating lucidity and his intransigence.¹⁰

In “Frege’s Influence on Wittgenstein,” Erich Reck factors this definition of platonism into three heuristic metaphors:

(i) numbers and other mathematical entities are “abstract objects” which exist “in their own right;” (ii) in mathematics we “describe” these objects, i.e., we talk about them as members of a “mathematical realm;” and (iii) the task of the mathematicians is to “explore” this realm, i.e., to find out what is “objectively the case” in it.¹¹

From his analysis of the work of Paul Benacerraf, Erich Reck concludes that Platonism remains a popular philosophical position for mathematicians, because it has set its agenda on the philosophy of mathematics. For the last hundred years, Platonism has defined the

⁹. Paul Edwards, ed., *The Encyclopedia of Philosophy* (New York: Macmillan: 1967)

¹⁰. (Barker 1967, 529) Emphasis added.

¹¹. Erich Reck, “Frege’s Influence on Wittgenstein” in William W. Tait ed., *Early Analytic Philosophy* (Chicago: Open Court, 1997), 126.

four explanatory goals of a philosophical account of mathematics. They are: (i) the nature of mathematical entities, (ii) the meaning of mathematical expressions, (iii) the objectivity of mathematical propositions, and (iv) the possibility of mathematical knowledge.¹² Offering an alternative explanation of these four points is not sufficient. A true challenge to Platonism requires redefining the agenda in philosophy of mathematics. Wittgenstein's grammatical picture successfully accomplishes this.

Wittgenstein rejects Platonism's three metaphors. For him, (i) mathematical entities are not genuine objects, much the less abstract or self-subsistent. (ii) Mathematical propositions are not descriptions. Mathematics does not describe mathematical entities. And (iii) mathematical calculations are not explorations of some esoteric 'mathematical realm', but searches in well-defined, logical spaces. However, Wittgenstein's objection goes beyond the mere rejection of these metaphors. It offers a new agenda for the philosophical investigation of mathematics.

Wittgenstein substituted the Platonists' basic questions with his own. The nature of mathematical entities stopped being an ontological issue and became a grammatical one. Since mathematical entities are not objects, but grammatical categories, questions about their independent subsistence become meaningless. Their existence becomes bound to the calculus they belong to. Calculations are not descriptions. Accordingly, questions about their descriptive powers or their truth are largely mistaken. Trying to explain how mathematical statements describe the abstract world of mathematics is nonsense. The relevant philosophical question is 'How do mathematical propositions rule calculations?' As rules, mathematical propositions are neither true or false. As calculations, they are either correct or incorrect. Wittgenstein substitutes for the Platonist's 'truth' of mathematical

¹². Ibid. 128.

propositions in terms of their accurate description of mathematical reality, the correctness of calculations.

2. Searches and Explorations

A mathematician is a blind man in a dark room
looking for a black cat which isn't there.
Charles Darwin.

Wittgenstein's objection to the Platonists' account of calculation is the conflict between two metaphors. For Platonists, mathematical calculations are excursions, while Wittgenstein sees them as searches.

Und 'suchen' muß immer heißen: systematische suchen. Es ist kein Suchen, wenn ich im unendlichen Raum nach einem Goldring umherirre. [PR §150 p. 165]

And 'search' must always mean: search systematically. Lost and wandering in infinite space looking for a gold ring is not searching. [PR §150 p. 175]

The view of calculation as exploration is closely linked to that of an independent mathematical realm. In geography, a successful exploration consists of the discovery and faithful description of a new territory. Similarly, for platonists, the discovery and faithful description of new mathematical territory makes for mathematical success. Frege expressed this position in his late article "Thoughts" (1918-19),

Thus for example, the thought we have expressed in the Pythagorean Theorem is timelessly true, true independently of whether anyone takes it to be true. It needs no owner. It is not true only from the time when it is discovered; just as a planet, even before anyone saw it, was in interaction with other planets.¹³

For Wittgenstein, the metaphor of calculations as explorations is misleading, because it suggests that mathematical truth is independent of the calculation. It suggests that the success of a mathematical calculation depends on whether it produces a correct or incorrect result. It suggests that a calculation is correct if it reveals a mathematical truth. Instead, in

¹³. Gottlob Frege, *Collected Papers*, ed. by Brian McGuinness (Oxford: Basil Blackwell, 1984), 363.

mathematics, the truth of the result depends on the success of the calculation. For example, to correctly calculate the product of 25 by 16, arriving at the correct result – 400 – is not sufficient. 400 is the product of 25 by 16, precisely because it is the result of multiplying these numbers, not the other way around. The product of two numbers is simply the result of correctly multiplying them. The calculation is not correct, because it produces the correct result. The result is correct, because the calculation is successful. This holds for the solution of any mathematical problem, for example, proving a theorem in a formal system. A proof is not correct, just because it results in a theorem. Instead, the result of a proof is a theorem, if the proof is correct.

In §150 of the *Philosophical Remarks*, Wittgenstein writes:

Was das Verständnis erschwert, ist die falsche Auffassung, als wäre die *allgemeine* Lösungsmethode nur ein – nebensächliches – Hilfsmittel zur Erhaltung von Zahlen, die die Gleichung befriedigen. Während sie an sich ein Aufschluß über das Wesen (die Natur) der Gleichung ist. Sie ist – wieder – kein nebensächliches Hilfsmittel zum Finden einer Extension, sondern Selbstzweck. [p. 163]

What makes understanding so difficult is the misconception of the *general* method of solution as only an –incidental– expedient for deriving numbers satisfying the equation. Whereas it is in itself a clarification of the essence (nature) of the equation. Again, it isn't an incidental device for discovering an extension, it's an end itself. [p. 173]

This does not imply that arithmetical – or any other kind of mathematical – propositions are empirical generalizations. Mathematical propositions do not report on the way people perform calculations. The correctness of a calculation does not depend on the performance circumstances. The correctness of a multiplication, for example, is not contingent on the way one multiplies, but on the rules of multiplication. When Wittgenstein talks about the correctness of a calculation, he is not referring to any action, particular or general. Mathematics does not say anything about what goes on in the mind of mathematicians while calculating. Mathematics is not the psychology or sociology of calculation. It is its gram-

mar. Michael Young explains,

It would indeed be paradoxical to suppose that we can base *a priori* judgements on the observed outcomes of activities performed on paper or ‘in our heads’ if the empirical judgements that reflect the observations in question served directly as evidence for the arithmetical judgement that we make. [It seems clear that one performs the calculation, for example ‘ $79 + 86 = 165$ ’] by first producing a symbolic representation of the mathematical [calculation] concept ‘the sum of 79 and 86’, (which we do probably by writing down the numerals ‘79’ and ‘86’, one atop the other or one after the other with an addition ‘+’ sign in between), then performing certain activities in which the symbols figure as elements (writing down a ‘5’ at the bottom of the right-most column, carrying the ‘1’, etc.), and by then noting that the outcome of our activities is the numeral string ‘165’ which we take to represent the number that is the actual sum we have calculated.¹⁴

Certainly, this activity is not mere tinkering, simply happening to yield a certain outcome. A clear and precise system of rules governs calculation. These rules, moreover, do not just provide recommendations as to how to best perform the activity in question. More specifically, they serve to define the activity itself. The rules of addition, for example, specify what it is to calculate a sum. Failing to follow the rules is failing to calculate. A miscalculated addition is no addition at all. An incorrect calculation is no calculation at all. To note the activity’s outcome is not merely to observe a certain string of numerals turning up at the bottom of a column. Rather, it is to notice that the procedure of calculation defined by the rules yields this numeral string as its only and inevitable outcome.

Natural language reflects these circumstances in its talk of ‘calculation’. After doing everything just as it ought to be done, it is legitimate to claim having performed *the* calculation. It is possible to distinguish among different performances of the activity or different agents performing the calculation. So long as these performances are properly executed, they all qualify as performances of *the* calculation. A mathematical proposition does not merely express the observation that a certain numerical expression turns up at the

¹⁴. Michael Young, “Kant on the Construction of Arithmetical Concepts” *Kant-Studien* 73 (1982), 25.

bottom of the column or following the “=” sign. Instead, it expresses that such a numerical expression turns up as *the* outcome of *the* calculation.

Judging the result of a calculation, however, does involve the judgement of matters external to the calculus. In Michael Young’s words, “as these comments suggest, when we ground an arithmetical judgement on the performance of a calculation we do, in fact, presuppose the truth of a number of empirical judgements.”¹⁵ For example, arithmetical calculation requires successfully recognizing the symbols in the calculation as belonging to their proper number-types. However, these judgements are not part of the content of mathematical propositions. They do not provide evidence for the truth of the mathematical proposition. The role these judgements play is simply supporting the judgement that one has performed the calculation correctly and that the numerical expression generated is, therefore, *the* outcome of *the* calculation in question.

Making mistakes in these matters is possible, of course. It is possible to misidentify a certain character as a ‘6’, when, in fact, it is an ‘8’. Accidentally skipping a numeral, carrying improperly, and so on, are all possible mistakes. In these cases, the numerical expression appearing at the end of the performance of the activity may not be the correct one. However, the calculation has not yielded the wrong answer. Correct calculations do not yield wrong answers. Rather, sometimes, being mistaken in thinking to have correctly performed the relevant calculation is possible. However, no arithmetical equation can express this fact. Only an appropriate genuine proposition can. ‘I have performed the addition of 64 and 78, and the result was 142’ is a genuine proposition. It does not amount to the mathematical proposition ‘ $64 + 78 = 142$ ’. The first one may be false. It is possible to miscalculate the addition. The second one, by contrast, is necessary. Furthermore, it

¹⁵. Ibid. 24.

determines the truth (or falsity) of the previous one, not the other way around. Attempting to add 64 and 78, but producing a numeral ‘168’ as outcome, the miscalculation is clear, precisely because $64 + 78 = 142$.

In committing a mathematical error, or making a mathematical conjecture, the object of false or hypothetical belief is not a mathematical proposition, but a genuine proposition reporting on the (mis)calculation.¹⁶ It is possible to believe that ‘ $625 + 1223 = 1818$ ’ is true after miscalculating, for example. But then the proposition ‘ $625 + 1223 = 1818$ ’ occurs *de dicto*, not *de re*. The mistaken belief is not about some matters in the abstract mathematical realm. It is about the expression ‘ $625 + 1223 = 1818$ ’. Namely, it is the belief that it results from a correct calculation. It is not the belief that it expresses a true mathematical proposition.

Just as mathematical success is the success of calculation, mathematical mistakes are the result of miscalculation. In consequence, mathematical calculations, including proofs, are not made of propositions, but of expressions. Their correctness is purely grammatical.

B. Correct Calculations and Correct Results

1. Calculations as Transitions

<p>Aber <i>entshuldigen</i> Sie! . . . Im Kalkül interessiert man sich ja immer dafür, was heraus- kommt: wie seltsam? Das kommt da heraus – und dort das! Wer hätte das gedacht!</p>	<p>But <i>excuse</i> me! . . . In the calcu- lus, we are always interested in the result. How strange! This comes out here – and that there! Who would have thought it?</p>
<p>PR Appendix II. p. 319</p>	<p>PR. Appendix II. p. 332</p>

Wittgenstein established that calculation solves all mathematical problems. However, the correctness of the solution and the correctness of the calculation are not equivalent. For any

¹⁶ The object of false belief is its ‘explicitly grammatical counterpart’ as defined in chapter 7.

mathematical question, a correct calculation always gives the correct solution. However, it is also possible for incorrect calculations to yield the right solution to the problem, perhaps by accident. If the calculation is correct, then the solution is correct. But, in many cases, giving the correct solution does not guarantee the calculation's correctness.

The correct performance of an algorithmic mathematical calculation entails the correctness of its result. A correct calculation always yields the correct result. A well performed multiplication produces the correct product of its factors as result. The result of drawing a circle correctly must be a circle. Also, proving a formula correctly fully justifies calling it a theorem. However, the converse is not true. Sometimes, incorrect calculations also yield correct results. Consider the aforementioned examples of calculating the product of two numbers, drawing a circle and proving a theorem in a formal system. In these cases, a correct result does not always entail a correct calculation. For a proof to be correct, it takes more than the result to be a theorem. An incorrect multiplication may also produce the correct product. It is, in principle, possible to draw a circle free hand. In all these cases, it is possible to obtain the correct result without performing the correct calculation.

However, for Wittgenstein, the calculation entirely determines the correctness of the solution. Wittgenstein insists that calculations are transitions between expressions. He is never clear as to whether the transition happens from the calculation expression to the solution expression or from the expression of the mathematical question to the complete mathematical proposition. Consider the mathematical question ' $37 \times 48 = ?$ '. Its solution is '745'. Wittgenstein does not say whether the multiplication is the transition from ' 37×48 ' to '745' or from ' $37 \times 48 = ?$ ' to ' $37 \times 48 = 745$ '. In either case, the important point is that the multiplication is a transition between both expressions. It connects the calculation with its result. This is what the mathematical proposition expresses. ' $37 \times 48 = 745$ ' expresses

this transition. In contrast, finding the correct solution to the problem without calculating does not provide this connection.

The importance of the connection between calculation and result increases with the complexity of the calculation. In the basic cases, the correctness of result and calculation is the same. More complex cases require more than correct results. In the case of arithmetical multiplication, for example, multiplying small integers correctly involves nothing but getting the correct result. That is why children memorize the multiplication tables. Also, multiplying larger integers requires obedience to the rules of multiplication. To know that the calculation is correct requires more than a correct result. It requires something ensuring that the result was actually calculated, instead of guessed or copied. Multiplying small integers by memory and multiplying larger ones by algorithm are different sorts of calculations. Algorithmic calculations may be wrong and still yield correct results. Elementary school math students all over the world have complained about this for years. The correction of the calculation requires something else. It requires a sign or trace evidencing that some calculation yielded the result.

2. Traces

In some cases, the result of a process is not a material object, but a transition. In the case of processes like cooking or building, even when language is ambiguous, telling process from result is relatively easy. Man or nature may perform processes well or poorly, fast or slowly. Their results, as material objects, have material properties which the processes do not. A building may be tall or ugly, modern or old, etc. A food dish may be salty or sweet, well-done or overcooked, etc. However, not all processes produce a material object as result. Consider, for example, the process of growth. The result of growing is not an object, but a transition from one state to another. Examples of processes whose result is a transition are many. The result of movements, transformations, and similar processes is not an object, but

a transition. These cases have the extra difficulty that they do not only have a result, but also a *final result*. The final result of these processes is not the transition, but its final state. Consider the growth process that happened in my teen years. In those years I grew from 1.50 m to 1.78 m. The transition from 1.50 m to 1.78 m was the result of this process. The final result, on the other hand, was my being 1.78 m tall.¹⁷

Mathematical calculations are activities of this sort. Calculation is a process more like going or growing, than cooking or building. The result is not a material object, but a transition. The result of going from one place to another is the transition between being in one place and being in the other. The final result may be being at the arriving place, but the result of going there is the transition from the starting place. The absence of a material object as result does not imply that transitions do not produce anything material. In most cases, transitions leave a material trace. Consider, for example, walking from one edge of a beach to the other. The transition from one edge of the beach to the other is immaterial, but it leaves some marks behind. Since the sand was soft, the walking process could have left certain traces on it. Let us call these material byproducts the ‘trace’ of the process. As its name indicates, ‘traces’ trace the result back to the process that produced it. By the trace of someone’s steps on the grass, it is possible to connect transition with result, and starting point with the end. This trace shows that a person got to one side of the knoll from the other (result) by walking on the grass (transition). One also knows where exactly that person went (final result). Furthermore, it shows each step from one side to the other.

What counts in mathematics is what is written down. Symbols obviously interest even the intuitionist, who says that mathematics is not a science about symbols but about meanings – just as a zoologist might say, analogously, that zoology is not a science about the word “lion” but about

¹⁷. Ordinary language commonly presents the final result in such a way that it becomes explicit that it is the final result of a transition. In presenting the final result, one uses verbs like ‘becoming,’ ‘getting,’ and others lexical indicators like ‘now,’ ‘after that,’ which denote transition. I would say that, as a result of growing up during my teenage years, I became 1.78 m or that, as a result, I am 1.78 m now.

lions. But there is no analogy between mathematics and zoology in this respect. [WL *Philosophy for Mathematicians* 1932-33 §11 p. 225]

In the case of calculations, the trace is the group of symbols written while performing the calculation. For example, when adding 345 and 786, the trace may be something like this:

$$\begin{array}{r} 11 \\ 345 \\ +786 \\ \hline 1131 \end{array}$$

The result is 1131, but the trace is the complete group of numerals, lines and ‘carries’ written above. They show not only the calculation performed (the addition of 345 and 786) and its final result (1131), but also the intermediate steps taken. The ‘1’ above the ‘4’ in ‘345,’ says that adding five and six first resulted in a number greater than nine, but less than twenty. The final result is not, properly speaking, part of the trace. Even though the numeral ‘1131’ occurs in the trace of the addition, the actual final result is not a numeral, but a number: 1131.¹⁸

IV. Conclusion

Wittgenstein distinguishes between mathematical problems and problems of mathematical investigation. Mathematical problems are tasks of calculation that demonstrate the correctness of calculation propositions. By contrast, problems of mathematical investigation are misguided attempts at calculating what cannot be calculated. They are either attempts to calculate non-mathematical propositions or ungrammatical pseudo-propositions.

Every mathematical problem asks if a calculation proposition is correct. A calculation can solve a mathematical problem, if a general method of solution exists. Calculations are not explorations of some esoteric ‘mathematical realm’, but searches in

¹⁸. In these cases, it is feasible to say that the result is the *meaning* of the trace, and the final result is the *meaning* of only a part of it.

well-defined, logical spaces. Checking the truth of a calculation proposition is looking for it among the rules defining the calculation.

Mathematical propositions are true in a completely different sense than other propositions. In the case of a calculation proposition p , p is true means that p is correct, according to the calculus. Since a calculation proposition connects a calculation to its result, a calculation proposition is correct if and only if it connects a calculation with its correct result. The calculation ultimately bears the solution of a mathematical problem. In consequence, it is more proper to talk about mathematical *correctness* than mathematical *truth*. Truth and falsity do not apply to mathematics, only correctness and incorrectness. Mathematical truth is not independent of calculation. A calculation is not correct, because it reveals a mathematical truth. Instead, mathematical truth is the correctness of calculation.

Calculations are not events. Mathematics is not a natural science. It is not the psychology or sociology of calculation. Mathematical propositions are not empirical generalizations. They do not report on the way people perform calculations. A mathematical proposition does not merely express the observation that a certain expression turns up at the end of the calculation. Instead, it expresses that such expression turns up as *the* outcome of *the* calculation. It connects the calculation with its result. In consequence, the solution to a mathematical problem involves more than just producing a correct result. It requires a connection between problem and solution. It requires something ensuring that the result was actually calculated. A sign, tracing the result back to the calculation that produced it, is also necessary.

Every calculation is a transition from one expression to another, according to a rule. The ‘forming’ of every step in the calculation involves the comparison of expressions with some paradigm representing the scheme of the transition.

Chapter 4

Mathematical Application [*Anwendung*]

I. Introduction

Section III of the second part of the *Philosophical Grammar*, ‘Foundations of Mathematics’, [*Grundlagen der Mathematik*] focuses on the notion of application [*Anwendung*]. As the section’s title suggests, Wittgenstein primarily explores the role application plays in the foundations of mathematics. The application of mathematics requires no foundation. Using formal tools for preparing mathematics for its application (proofs of relevance, proofs of consistency, and formal interpretations) is superfluous and misguided. Such efforts are superfluous, because preparing a calculus for application is unnecessary. The application “takes care of itself.” They are misguided, because calculation can only solve mathematical problems, and a calculus’ applicability is not a mathematical problem. Instead, the quest for a foundation for mathematics is a philosophical problem. Calculation cannot yield a foundation for mathematics, because mathematics is “well enough grounded in itself” [*genug in sich selbst begründet*. PG §15 p. 600 (p. 306)].

Calculations cannot solve anything but mathematical problems. In consequence, mathematical calculations can only solve mathematical problems. Even though mathematical calculations play a central role in the solution of practical problems, they do not offer, entail or justify predictions about affairs outside the calculus. Mathematical calculations are used all the time to solve practical problems. However, calculations cannot answer empirical questions. Mathematical calculations provide the same sort of solutions to practical and mathematical problems. They provide a grammatical rule whose applications are propositions

either inside or outside the calculus. This explains their capacity to help us say things about real objects and their properties.

The logicians' accounts of mathematical application, like those of Frege, Russell and Ramsey, treat calculations as universal propositions entailing predictions about the world. Wittgenstein challenges the logicians' view that the solution of a practical problem involves inferring a prediction about the world from mathematical calculations. For Wittgenstein, the question 'How it is possible to infer a genuine proposition about the world from a mathematical calculation or statement?' is nonsensical. Mathematical propositions are not genuine propositions. They belong to a different logical space than propositions about genuine objects and events. Inference among mathematical propositions is calculus-bound. A mathematical proposition relates inferentially only to propositions within the same calculus. It does not entail or is entailed a proposition outside the calculus. It is impossible to infer a genuine proposition about something from a mathematical proposition about nothing. "The calculation is only a consideration of logical forms, of structures, and of itself can't yield anything new" [*Die Rechnung ist nur eine Betrachtung der logischen Formen, der Strukturen, und kann an sich nicht Neues liefern.* PG Pt. III §15 p. 604 (p. 307)]. The calculation (or associated mathematical statement) is not a premise in the solution of practical problems.

Wittgenstein does not erase the distinction between pure and applied mathematics. He challenges its interpretation. Logicians mistakenly view mathematics as a machine or tool made in preparation for some predefined use. "Here it is a matter of our concept of *application*. – We have an image of an engine which first runs idle, and then drives a machine." [*Hier handelt es sich um unsern Begriff der Anwendung. – Man hat etwa die Vorstellung von einem Motor, der erst leer geht, und dann eine Arbeitsmaschine treibt.* PG §15 pp. 604, 606 (p. 309)] For Wittgenstein, mathematical calculations and grammatical rules share the

same application. Their application is the construction and transformation of propositions. Since calculations themselves are rules of the calculus, they apply to themselves. Mathematical calculations are their own *internal* applications. The construction and transformation of propositions inside the calculus constitute its internal application. They also apply externally to genuine propositions or to propositions from other calculi. Because neither application is part of the calculus, application is not part of mathematics either. From the perspective of the calculus, pure and applied mathematics are not different.

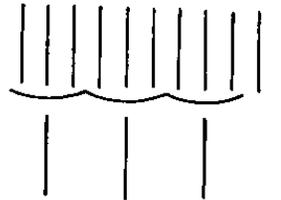
Application does not found the calculus. In solving practical problems, the calculation neither becomes empirical nor acquires some extra *reality* it lacked before. Attempts at formalizing the conditions of calculus application only result in extra calculi. However, the extra calculi do not provide a foundation for the original one. It does not make sense to make preparations for the application of arithmetics or any other mathematical calculus. The calculus is its own application. If the calculus exists, then at least one application of it exists: *itself*.

For this reason, Wittgenstein rejects the traditional view of consistency proofs. Using a calculus does not require a proof of its consistency. If consistency were necessary for the application of any calculus, it would not be a syntactic property. No calculation could prove it. On the other hand, if consistency meant only the absence of contradictions, consistency proofs would have no effect on the calculus. Thus, proving formally the applicability of a calculus is misguided and doomed to failure.

II. Calculations' Role in the Solution of *Practical* Problems

<p>Hier kann man nun sagen: Die Arithmetik ist ihre eigene Anwendung. Der Kalkül ist seine eigene Anwendung. PG §15 p.608</p>	<p>At this point we can say: arithmetic is its own application. The calculus is its own application. PG §15 p. 310</p>
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Wittgenstein explains the role mathematical calculations play in the solution of practical problems, and the apparent difference between pure and applied mathematics in section III, 'The Foundations of Arithmetic' [*Die Begründung der Arithmetik*], of the *Philosophical Grammar*. In the relevant passages of this section, Wittgenstein considers set theory [§15 p. 606 (p.309)] and geometry [§17 pp. 626, 628 (pp. 319, 320)] in addition to arithmetic. However, his considerations of such cases are completely analogous to those of arithmetic. Since the rest of the dissertation has concentrated on arithmetic examples, this chapter focuses on the following passage from §15 of the *Philosophical Grammar*:



Angenommen, mit dieser Rechnung wollte ich folgende Aufgabe lösen: Wenn ich 11 Äpfel habe und Leute mit je 3 Äpfeln beteilen will, wieviele Leute kann ich beteilen? Die Rechnung liefert mir die Lösung 3. Angenommen nun, ich vollzöge alle Handlungen des Beteilens und am Ende hätten 4 Personen je 3 Äpfel in der Hand. Würde ich nun sagen, die Ausrechnung hat ein falsches Resultat ergeben? Natürlich nicht. Und das heißt ja nur, daß die Ausrechnung kein Experiment hat.

Es könnte scheinen, als berechtigte uns die mathematische Ausrechnung zu einer Vorhersagung, etwa, daß ich 3 Personen werde beteilen können und 2 Äpfel übrigbleiben werden. So ist es aber nicht. Zu dieser Vorhersagung berechtigt uns eine physicalische Hypothese, die Außerhalb der Rechnung steht. Die Rechnung ist nur eine Betrachtung der logischen Formen, der Strukturen, und kann an sich nicht Neues liefern. [p. 602] [p. 602. Cf. Also PR §111 p. 122]

Suppose I wish to use this calculation [*Rechnung*] to solve the following problem: if I have eleven apples and want to share them among some people in such a way that each is given three apples how many people can there be? The calculation supplies me with the answer 3. Now suppose I were to go through the whole process of sharing and at the end 4 people each had 3 apples in their hands. Would I then say that the calculation [*Ausrechnung*] gave a wrong result? Of course not. And that of course means that the calculation [*Ausrechnung*] was not an experiment.

It might look as though the mathematical calculation [*Ausrechnung*] entitled us to make a prediction, say, that I could give three people their share and there will be two apples left over. But that isn't so. What justifies us in making this prediction is an hypothesis of physics; which lies outside the calculation. The calculation is only a study of logical forms, of structures, and of itself can't yield anything new. [p. 307 Cf. also PR §111 pp. 132, 133]

The calculation's role in the solution of this problem intrigued Wittgenstein. Most of all, he was interested in the relationship between calculation and physical prediction. In this passage, Wittgenstein illustrated the philosophical differences between calculation and experiment by distinguishing numbers as *solutions* of practical problems and *results* of a calculation. In cases like Wittgenstein's example, the result of the calculation is also the solution to the problem. Number '3' is both the solution to the problem and the result of the calculation. However, Wittgenstein distinguishes these two roles of '3', while observing their close kinship.

A calculation's result may also be the solution of a non-mathematical problem, because it establishes the possibility of making non-mathematical predictions. This particular example predicts that to distribute twelve apples to at most *three* persons is possible. As such, the prediction may be true or false. For this reason, it makes sense to "suppose now, that I carry through the distribution and at the end there are four persons, each one with three apples in their hand." It is possible to imagine the prediction's falsity. It is possible to conjecture that the solution of a problem will fail. The calculation provides for this possibility. As such, the mathematical calculation cannot guarantee the prediction's success.

The calculation *alone* cannot produce anything new. The prediction's guarantee must lie outside the calculation.

Predictions are physical hypothesis outside the calculus. On the other hand, results belong to the calculus. A well-defined border separates these two logical spaces. The reason why Wittgenstein frames this distinction in the headline "The foundation of mathematics in which it is prepared for its applications" remains unexplained.

A. Wittgenstein's Criticism of The Logicians' Account of Mathematical Application

1. The Logician's Puzzle

In one sense there is no science of applied mathematics. When once the fixed conditions that any hypothetical group of entities are to satisfy have been precisely formulated, the deduction of the further propositions, which also will hold respecting them, can proceed in complete independence of the question as to whether or not any such group of entities can be found in the world of phenomena.

A. N. Whitehead¹

Logicians' explanation of mathematical application involves inferring physical predictions from mathematical calculations and propositions. The logicians would explain a case of mathematical application like that in *Philosophical Grammar* §15 through the inference of a physical prediction like 'If I have eleven apples and want to share them among some people in such a way that each is given three apples, there can be *three* people' from the mathematical equation $11 / 3 = 3$. Justifying this inference challenges the logicians's view, because, for them, mathematical propositions like ' $11 / 3 = 3$ ' are analytic, while propositions like 'If I have eleven apples . . . ' are synthetic. Therefore, they must explain the possibility that an analytic mathematical proposition entails a synthetic one. In order to solve practical

¹. Whitehead, Alfred North: "Foundations of Mathematics" article for the *Encyclopædia Britannica* (Chicago, 1988) <http://www.britannica.com>.

problems, mathematical propositions must “both be true regardless of fact and also imply a truth about . . . observable objects.”² Alice Ambrose presents this puzzle as follows.

How then can one account for the harmony between the two different areas of logic and empirical fact? How is it that we can apply arithmetical calculations to physical objects, or trigonometrical calculations to physical lines and angles? Is there a genuine mystery here or only a gratuitous puzzle?³

Two widespread myths about mathematics have stemmed from this puzzle. First, mathematics is a universal science. Mathematical propositions are universal. For example, equation ‘ $11 / 3 = 3$ ’ is a universal proposition about all possible additions. Second, there are propositions of so-called applied mathematics, which are neither genuine empirical propositions nor propositions of pure mathematics. Wittgenstein’s account of mathematical application challenges these two myths.

2. First Myth: Mathematics is the Most Universal of Sciences

Logic is concerned with the real world just as truly as zoology, though with its more abstract and general features.

Bertrand Russell⁴

Logicians mistakenly approach mathematics as the most abstract and universal of sciences. For Wittgenstein, mathematics is not about everything – as the logicians maintained, but about nothing. The universality of mathematics is no guarantee for its many applications, because mathematics is not more universal than its applications.

For Wittgenstein, mathematical operations are not universal. For example, mathematical addition is not an abstract generalization of all possible additions. Logicians believed

2. Alice Ambrose, “Some Questions in Foundations of Mathematics” in Stuart Shanker: *Ludwig Wittgenstein. Critical Assessments. Volume Three: from the Tractatus to Remarks on the Foundations of Mathematics: Wittgenstein on the Philosophy of Mathematics*. (London: Croom Helm, 1986) 204. Originally published in *Journal of Philosophy* vol. 52 (1955) 197-213.

3. Ibid.

4. Bertrand Russell, *Introduction to Mathematical Philosophy* (1920) 169.

that whatever arithmetic says about addition applies to all possible additions. They thought that was the reason mathematics had such diverse applications, from social behavior to elementary particle physics. For them, mathematical propositions have an implicit hypothetical antecedent expressing their conditions of application. ‘ $4+4=8$ ’, for example, means that ‘if ‘+’ is an operation defined on the extension of concept Φ such that ‘+’ obeys all the basic rules of arithmetical addition, according to the equivalence relation ‘=’, then ‘ $4 \Phi + 4 \Phi = 8 \Phi$.’” According to the logicians, the arithmetical proposition ‘ $4+4=8$ ’ says that, for the proper sort of objects, adding four of them to another four results in four objects of that sort. For example, logicians believe that, since apples are among the proper sort of objects, ‘ $4 + 4 = 8$ ’ entails the *more particular* proposition that ‘if one has four apples and adds them to other four apples, one will have eight apples’. For the logicians, ‘4 apples + 4 apples = 8 apples’ is an application of ‘ $4 + 4 = 8$ ’. The primary difference between the proposition about apples and the purely mathematical one is that the latter is a verifiable, physical hypothesis. An experiment can verify that adding four apples to four apples results in eight apples. In general, for the logicians, propositions in the arithmetics of natural numbers entail genuine propositions about apples. The application of the arithmetic of natural numbers to apples justifies this inference.

3. Wittgenstein against the Universality of Mathematics

For Wittgenstein, mathematics is not a universal science. Mathematical statements are not universal. The arithmetical proposition ‘ $4 + 4 = 8$ ’ is not about every addition. It is about arithmetical addition only. Furthermore, showing that apples are objects of the proper sort does not justify inferring ‘4 apples + 4 apples = 8 apples’ from ‘ $4 + 4 = 8$ ’. Adding the term ‘apple’ to a proposition in pure arithmetic does not create a new proposition of applied arithmetics about apples.

Man muß sich aber davor hüten zu glauben “4 Äpfel + 4 Äpfel = 8 Äpfel” ist die konkrete Gleichung, dagegen $4 + 4 = 8$ der abstrakte Satz, wovon die erste Gleichung nur eine spezielle Anwendung sei. So daß zwar die Arithmetik der Äpfel viel weniger allgemein wäre, als die eigentliche allgemeine, aber eben in ihrem beschränkten Bereich (für Äpfel) gälte. – Es gibt aber keine “Arithmetik der Äpfel”, denn die Gleichung $4 \text{ Äpfel} + 4 \text{ Äpfel} = 8 \text{ Äpfel}$ ist nicht ein Satz, der von Äpfeln handelt. [PG §15 p. 604]

But we must be aware of thinking that “4 apples + 4 apples = 8 apples” is the concrete equation and $4 + 4 = 8$ the abstract proposition of which the former is only a special case, so that the arithmetic of apples, though much less general than the truly general arithmetic, is valid in its own restricted domain (for apples). There isn’t any “arithmetic of apples”, because the equation $4 \text{ apples} + 4 \text{ apples} = 8 \text{ apples}$ is not a proposition about apples. [PG §15 p. 308]

Wittgenstein claims that “there isn’t any *arithmetic of apples*,” meaning that no third realm of applied mathematics exists between mathematics and the real world. It does not make sense to talk about propositions like ‘4 apples + 4 apples = 9 apples’ as being neither mathematical nor genuine. No middle ground between genuine and mathematical propositions exists.⁵

4. Letters and Schemes.

Dispelling the myth that mathematical propositions are universal requires reinterpreting the role of quantification and variables in mathematics. Wittgenstein does this in sections XIII and XIV, Part II of *Philosophical Grammar*, by focusing on the syntactic role of letters. In paragraph 150, Wittgenstein described the three possible functions of a letter in mathematics: (1) as general constant, (2) as unknown and (3) as marker for a blank space. In the first case, the letter belongs to the language of the calculus. In the other two cases, it is an external element. In either case, every letter in a calculation statement is a general constant. For Wittgenstein, there is no significant difference between general constant and

⁵. For Wittgenstein, ‘4 apples + 4 apples = 8 apples’ is not a genuine proposition about apples. It is the mathematical proposition $4 + 4 = 8$ expressed in terms of apples instead of numbers.

‘universal’ variables. All mathematical variables in a calculation statement are universally quantified, not only those under the explicit scope of a quantifier. However, in mathematics, ‘universal quantification’ does not mean universality in the platonist sense. Wittgenstein also rejected the traditional interpretation of the universal quantifier in mathematics. In note I to paragraph 150 of the *Philosophical Remarks*, he wrote:

Dieses zeichen ‘(x)’ sagt aber gerade das Gegenteil dessem, was es in den nicht mathematischen Fällen sagt . . . nämlich gerade, daß wir die Variable in dem Satz als *Konstante* auffassen sollen. [PR §150 n. I, p. 164]

But this sign ‘(x)’ says exactly the opposite of what it says in non-mathematical cases . . . i.e. precisely that we should treat the variables in the proposition as *constants*. [PR §150 n. I, p. 174]

For Wittgenstein, letters in mathematics and logical notation have radically different meanings. Disregarding this difference produces the mistaken idea that variables may occur unbound. Logical formulae represent the logical form of genuine propositions. Since genuine propositions can be general, the ability to express generality in the logical formalism is necessary. This is the role of letters in logical formalism. However, mathematical formulas do not represent the logical form of genuine propositions. Accordingly, their letters have a different role. Talk about *all* the numbers, for example, may suggest that mathematical propositions are universal, but mathematical generality is of a different sort. Mathematical generality joins totality and necessity in a single notion. “For in mathematics ‘*necessary*’ and ‘*all*’ go together. (Unless we replace these idioms throughout by ones which are less misleading.)”⁶ Mathematical modality is not that of possible / necessary or universal / particular, but sense / nonsense. Mathematical modality is *syntactic*.

According to the previous analysis, a mathematical formula has three different interpretations depending on the syntactic role of its letters. This classification corresponds

⁶ PR §150

to a division in the possible meaningful questions about such expressions:⁷

(1) If all its letters are constants in the calculus, the formula is a calculation proposition.

It makes sense to ask whether the formula is correct or not.⁸ In the case of equations, the answer to this question depends on whether or not the rules of the calculus allow for each side of the '=' sign to transform into the same expression.⁹

(2) If at least one of the letters expresses an unknown and no letter marks a blank space, then the formula is not a proposition but a scheme. Asking if the equation is solvable makes sense. The equation is solvable only if the replacement of the unknown letter for a general constant (not necessarily a letter) results in a true proposition.

(3) If at least one of the letter marks a blank space, then the expression is incomplete. It makes sense to ask if it is syntactically permissible. An incomplete expression is syntactically permissible if it is possible to construct a well-formed formula by filling its blank spaces.

Wittgenstein adopts this classification to dispel the myth that mathematics has different levels of generality. Expressions with letters are no more general than expressions with other mathematical constants, like numerals. In a footnote to paragraph 150 of the *Philosophical Remarks*, Wittgenstein notes,

Ich habe noch zu wenig betont daß $25 \times 25 = 625$ auf *genau* derselben Stufe und von *genau derselben Art* ist wie $x^2 + y^2 + 2xy = (x+y)^2$. [PR §150 n.1, p. 164]

I still haven't stressed sufficiently that $25 \times 25 = 625$ is on *precisely* the

7. In strict sense, it is not any mathematical *equation*, but any expression *in the form of* an equation, since it may turn out to be that the given expression is not a genuine equation.

8. Unless, of course, the equation contains other letters, which are not constant variables.

9. PR 154

same level as and of *precisely the same kind* as $x^2 + y^2 + 2xy = (x+y)^2$. [PR §150 n. 1, p. 174]

Wittgenstein illustrates this distinction with the expression ' $a+(b+c)=(a+b)+c$ '. This formula expresses both an algebraic proposition and the law of associativity for arithmetical addition. As an algebraic proposition, its letters are general constants. In the law of associativity, they are schematic letters, and the expression is a scheme.

The law of additive associativity in arithmetic ' $(a+b) + c = a + (b+c)$ ' seems to express a general property of all numbers or all additions. However, it only appears to be general. Distinguishing cases or instances from applications is crucial to understanding the apparent generality of mathematical propositions. Consider the following four propositions:

- (1) $(a+b) + c = a + (b+c)$
- (2) $(3+4) + 6 = 3 + (4+6)$
- (3) All camels are herbivores.
- (4) My camel is a herbivore.

Mathematical propositions as grammatical rules are *applied*. Genuine generalizations have instances. (2) is an application of (1), while (4) is an instance of (3). The truth of genuine generalizations like (3) rests on induction from particular cases like (4). The truth of grammatical claims like (1) does not depend in anyway on that of propositions like (2). (2) in no way confirms (1). Even if proposition (2) occurs in the proof of (1), it would not be a confirming instance. That is why mathematical induction is so different from genuine inductions as they feature in empirical science. In mathematics, instances do not exist. Hence, they cannot confirm any generalizations. Generality holds throughout all mathematical propositions. Proposition (2) is not less general than (1), even though it may seem that (2) is about particular numbers and (1) is about all of them.

In mathematics, generalization from particular cases is impossible. From a series of

known mathematical propositions, to come up with a more general proposition by generalizing into similar cases is impossible. Talk of ‘similar cases’ involves classifying the propositions through a general concept. However, mathematical concepts are disjunctions of their members. Bringing mathematical propositions together under a mathematical concept would result in a conjunction of the original propositions. A conjunction is not more general than the sum of its elements, though. In consequence, bringing mathematical propositions together under a mathematical concept does not result in a more general mathematical proposition. The resulting mathematical statement is neither less nor more necessary or general than its elements.

5. Wittgenstein and Ramsey on *Anwendung* and Interpretation

Wittgenstein entitled §15 of section II of his *Philosophical Grammar* ‘*Die Begründung der Arithmetik, in der diese auf ihre Anwendungen vorbereitet wird (Russell, Ramsey)*’ Anthony Kenny translated it for the English edition of *Philosophical Grammar* as ‘Justifying arithmetic and preparing it for its application (Russell, Ramsey)’. However, a better translation would be ‘The founding of arithmetic in which it [arithmetic] is prepared for its applications’. Kenny’s translation misleads in important ways. Kenny’s title suggests that Wittgenstein intends to take up the separate but related topics of justifying arithmetic and of “preparing arithmetic for its application” whatever that would be. The alternative translation reveals Wittgenstein’s specific target: Russell and Ramsey’s approach to the foundations of arithmetic, according to which a fully satisfactory foundation for arithmetic requires and includes provision to use logic to specify and control the application of arithmetic.

Evidence suggests that Wittgenstein directed his comments on this topic at Ramsey’s “The Foundations of Mathematics” (1925). First of all, ‘The Foundations of Mathematics’ is the section’s title. Second, Ramsey presented in “The Foundations . . .”

the identity theory Wittgenstein discusses in §16. Third, most of the quotations and examples Wittgenstein uses in this section come from Ramsey's article. For example, the use of apples as an example of the application of arithmetics originally occurs in Ramsey.

Mathematical application is not one of "The Foundations" central concerns. On a superficial reading, Ramsey's text does not say anything about the application of mathematics. The very first sentence of his introduction states that it pertains to "the general nature of pure mathematics" instead of *applied* mathematics. Nevertheless, in the very next paragraph, he states that any theory of mathematical concepts is "hopeless" if it only accounts for the meaning of mathematical terms in mathematical propositions.

. . . for these occur not only in mathematical propositions, but also in those of everyday life. Thus '2' occurs not merely in ' $2 + 2 = 4$ ', but also in 'It is 2 miles to the station'. . . Nor can there be any doubt that '2' is used in the same sense in the two cases, for we can use ' $2 + 2 = 4$ ' to infer from 'It is two miles to the Gogs via the station', so that these ordinary meanings of two and four are clearly involved in ' $2 + 2 = 4$ '.¹⁰

Ramsey's interest in his *Foundations of Mathematics* is the interpretation of mathematical formulae. In this article, perhaps for the first time in the history of logicism, Ramsey laid out a separation between formalism and interpretation. Ramsey distinguished between the formal elements of a theory – formal language, axioms and rules, and its interpretation – a mathematical domain with special, designated functions and relations defined over it. Ramsey makes it clear that systematizing the formalism alone, as Russell and Whitehead did in *Principia Mathematica*, is not sufficient for mathematics' foundation. For Ramsey, the foundations of mathematics require guaranteeing the proper interpretation, too. Described in contemporary terms, the proper interpretation of a formalism must have the property of satisfying the axioms of the formalism. Ramsey's concern for the interpretation of mathematical calculi focused on formalizing their application conditions. For Ramsey, the

¹⁰. Frank Ramsey, "The Foundations of Mathematics" p. 165

application of the formalism to the objects, functions, relations and propositions in its specification requires a proper interpretation of the formalism.¹¹

From Ramsey's standpoint, the application of natural number arithmetic requires the construction of a proper interpretation of every arithmetic function and operation. For Ramsey, applying natural number arithmetic to the distribution of apples, for example, requires a specific interpretation of the equivalence relation of *numerical equality* among groups of apples.¹² It is also necessary to specify interpretations for the mathematical operations of the arithmetic calculus. The operation of putting extra apples in a group of apples may interpret the arithmetical operation of addition, for example. The operation of taking apples from a group of apples may interpret the operation of subtraction, and so forth. Ramsey thought that, for the proper interpretation, these operations must follow the basic rules of the arithmetic operation. In the case of addition, for example, the interpretation must be associative, commutative, have a neutral element, et cetera.

For Wittgenstein, Ramsey's formal interpretation is a superfluous and misguided attempt at "preparing a calculus for its application." First of all, it is doubtful whether or not a full correspondence between pure, natural number arithmetic and some "arithmetic of apples" is possible. Furthermore, such correspondence would eliminate the putative difference between *pure* and *applied* arithmetic. If the interpretation of arithmetical operations on apples corresponded totally to the operations on natural numbers, both calculi would be as

11. Wittgenstein is not confusing the notions of interpretation and application. On the contrary, his criticism of Ramsey's position requires a clear separation between them. Wittgenstein criticizes Ramsey's idea that the provision of a proper formal interpretation of the calculus prepares it for its application.

12. Another important aspect of Wittgenstein's philosophy of arithmetic is its separation from *counting*. In Part II, section 21 of the *Philosophical Grammar*, Wittgenstein argues that, in arithmetic, counting is not a more primitive notion than numerical equality. One can tell that two groups are the same in number, without actually knowing which number this is. Wittgenstein takes it a step further to say that 'numerical equality' is not about numbers at all, even if its surface grammar suggests so. Talking about two groups having the same quantity of objects, misleads one into thinking that such thing as their common quantity exists.

general. It would not make sense to say that arithmetics with apples is a particular case of the more general arithmetics with numbers. Applied calculi are not less general than non-interpreted ones. Mathematics has no hierarchies of generality. They are all at the same level. If groups of apples followed the same arithmetical rules as natural numbers, they would *be* natural numbers themselves. “Calculation with apples is essentially the same as calculation with lines or numbers” [*Die Rechnung mit Äpfeln ist wesentlich dieselbe, wie die mit Strichen oder Ziffern*. PHG §15 p. 608 (p. 310)]. On the other hand, if elementary arithmetic and its interpretation on apples did not follow the same rules, they would be two different calculi. In either case, neither calculus could justify the other.

B. Wittgenstein’s Account of Mathematical Application

The main difference between Wittgenstein’s and the logicians’ account starts with the very interpretation of the non-mathematical problem. Taking a closer look at the relevant passage from §15 of the *Philosophical Grammar*, (or §111 of the *Philosophical Remarks*), three elements are distinguishable:

1. The problem [*Aufgabe*]: “If I have eleven apples and want to share them among some people in such a way that each is given three apples how many people can there be?” This is a general hypothetical question. Furthermore, it is a modal question asking what is *possible*. It asks for the number of people it is possible to give three apples from a group of eleven apples.

2. The solution [*Lösung*]: Wittgenstein makes it clear that the solution to the problem is the number 3, not that there can be 3 persons.

3. The prediction [*Vorhersagung*] that I could give three people their share of four apples, leaving two apples.

Wittgenstein’s view of mathematics as grammar addresses the relation between the

calculation and these three different elements. In the aforementioned passage, he explicitly says that the calculation *supplies* [*liefert*] the solution to the problem. It does not *entitle* [*berechtigen*] one in making the prediction. Calculation does not justify the solution of practical problems. Providing the solution to a non-mathematical problem is essentially different than justifying a non-mathematical prediction. Wittgenstein agrees that the calculation in his example says that $11 \div 3 = 3$ – calculation and equation are identical, but it does not predict that if one gave three people their share, “there will be two apples left over.” Still, it says that it is possible to share eleven apples among three people in such a way that each receives three apples. However, the latter is not a prediction, but a grammatical proposition.

The calculation says that it is possible to share eleven apples among three people in such a way that each receives three apples. The question ‘If I have eleven apples and want to share them among some people in such a way that each receives three apples how many people can there be?’ is a grammatical question. It asks what the largest possible number of people is that could receive three of eleven apples. This possibility is not physical. It is grammatical. “For the word “can” in that proposition doesn’t indicate a physical (physiological, psychological) possibility.” [*Denn das wort “kann” in diesem Satz deutet nicht auf eine physische (physiologische, psychologische) Möglichkeit.* PG §14 p. 596 (p. 304)]

The proper answer to the question “if I have eleven apples and want to share them among some people in such a way that each receives three apples, how many people can there be?” is not a universal statement about apples and their distribution. It is grammatical, that is, mathematical. Its solution must be an appropriate grammatical rule. The mathematical calculation provides this rule. The calculation gives the proper solution, because the question is grammatical. It is not a question about the necessary properties of apples or their distribu-

tion, but a grammatical question of what makes sense to predict. The answer is a rule for the use of the word ‘apple’. It says that the genuine proposition “I give three people their share and there will be two apples left over” is grammatically correct. The prediction “that I could give three people their share and there will be two apples left over” makes sense.

For Wittgenstein, the prediction about apples and their distribution is not the solution to the problem, but its *application*. The prediction that “I could give three people their share and there will be two apples left over” is not the solution of the practical problem. The solution to the mathematical problem precedes the physical prediction. The formulation of the prediction requires the mathematical calculation, because it is its application. The calculation provides the grammar of the genuine proposition. Formulating the prediction involves applying the calculation as a grammatical rule. Thanks to the calculation, the prediction that three people will receive their share of four apples with two apples left makes sense. Only in this sense do mathematical calculations apply to the solutions of non-mathematical problems. To apply a mathematical calculation means using it as a grammatical rule in the construction or transformation of propositions. In §107 of the *Philosophical Remarks*, Wittgenstein writes,

Die arithmetischen Sätze dienen, wie Multiplikationstabellen und dergleichen, oder auch wie Definitionen, auf deren beiden Seiten nicht ganze Sätze stehen, zur *Anwendung* auf die Sätze. Und auf etwas anderes kann ich sie ja sowieso nicht anwenden. (Ich brauche also nicht erst irgendwelche Beschreibung ihrer Anwendung.) [PR §107 p. 119]

Arithmetical propositions, like the multiplication table and things of that kind, or again like definitions which do not have whole propositions standing on both sides, are used in *application* to propositions. And anyhow I certainly can’t apply them to anything else. (Therefore I don’t first need some description of their application.) [PR §107 p. 129]

The application of a mathematical calculus to external propositions requires embedding the rules of one calculus into the grammar of the other. In particular, a calculus application to

genuine propositions requires embedding the calculus rules into the grammar of natural language. Thus, calculation rules become grammatical rules of natural language. The calculation $11 \div 3 = 3$ is not only a rule in the arithmetic of natural numbers, but also a grammatical rule of English. The construction of the English sentence, “I could give three people their share and there will be two apples left over” is one of its applications. According to Friedrich Weissmann’s notes, during a conversation at Schlick’s home on December 28, 1930, Wittgenstein said,

Was heißt es, einen Kalkül anwenden? . . . Man wendet den Kalkül in der Weise an, daß er die Grammatik einer Sprache ergibt. Dem, was die Regel erlaubt oder verbietet, entspricht dann in der Grammatik das Wort ‘sinvoll’ und ‘sinloss’. [PR Appendix II, section ‘Widerspruchsfreiheit’ p.309]

What does it mean to apply a calculus? . . . We apply the calculus in such a way as to provide the grammar of a language. For, what is permitted, or forbidden by the rules then corresponds in the grammar to the words ‘sense’ and ‘senseless’. [PR Appendix II, section ‘Consistency’ p.322]

In Wittgenstein’s example, the calculation provides the grammar of the genuine proposition about apples. The proposition makes the physical prediction. The calculation makes the prediction possible, but the prediction’s truth remains independent of the mathematical calculation. It requires further empirical testing. The calculation itself predicts nothing about apples.

In §17 of the *Philosophical Grammar*, Wittgenstein writes,

(Ein Satz, der auf einer falschen Rechnung beruht (wie etwa “er teilte das 3 m lange Brett in 4 Teile zu je 1 m”) ist unsinnig und das beleuchtet, was es heißt “Sinn haben” und “etwas mit dem Satz meinen”) [PG §17, p. 626]

(A statement based on a wrong calculation (such as “he cut a 3-metre board into 4 one metre parts”) is nonsensical, and that throws light on what is meant by “making sense” and “meaning something by a proposition”).

Applying a correct calculation results in a well-formed statement. Applying an incorrect calculation results in a nonsensical one. If the calculation is correct, as in the example on

§15, the proposition makes sense. If the calculation is incorrect, as in the example from §17, the proposition is nonsensical. In either case, the calculation does not make the genuine proposition true or false. The mathematical calculation has nothing to do with the truth of its application. It does not justify it. A correct calculation may yield a false non-mathematical proposition, as well as a true one. The truth of the non-mathematical proposition resulting from the application of the calculation is independent of the calculation.

III. *Anwendung* and the Foundations of Mathematics

Unter Anwendung verstehe ich das, was die Laut-verbindungen oder Striche überhaupt zu eine Sprache macht. In dem Sinn, in dem es die Anwendung ist, die den Stab mit Strichen zu einem *Maßstab* macht. Das *Anlegen* der Sprache an die Wirklichkeit.

By application I understand what makes the combination of sounds or marks into a language at all. In the sense that it is the application which makes the rod with marks on it into a *measuring rod*: *putting* language *up against* reality.

PR §54 p. 84

PR §54 p. 74

From the perspective of the calculus, pure and applied mathematics are not significantly different. Mathematical problems solve mathematical and non-mathematical problems in the same way. However, they are critically different from the perspective of *Anwendung*. In the solution of a problem in pure mathematics, the application of the calculation happens inside the calculus. The calculation is its own application. The solution of a non-mathematical problem applies the calculation to a genuine proposition outside the calculus.

A. The Autonomy of Mathematical Calculi

Der Kalkül setzt den Kalkül voraus. [PR §108 p. 120]

The calculus presupposes the calculus. [PR §109 p. 130]

Jede Rechnung der Mathematik ist eine Anwendung ihrer selbst und hat nur als solche Sinn. [PR §109 p. 120]

Every mathematical calculation is an application of itself and only as such does it have a sense. [PR §109 p. 130]

Hier kann man nun sagen: Die Arithmetik ist ihre eigene Anwendung. Der Kalkül ist seine eigene Anwendung. [PG §15 p. 608]

At this point we can say: arithmetic is its own application. The calculus is its own application. [PG §15 p. 310]

In section III of the *Philosophical Grammar*, Wittgenstein states that mathematical calculi are their own applications. Since mathematical calculations are also the rules of the calculus they belong to, they apply to themselves. Mathematical propositions are grammatical rules that govern the same language that expresses them. This latter sort of grammatical rule is common. For example, the statement ‘In English, the first word of every sentence is capitalized’ expresses a grammatical rule in the grammar of English. The rule applies to sentences in that language. In particular, it applies to the sentence that expresses it. However, neither the English sentence nor the grammatical rule apply to themselves. For that, the sentence would have to be autonomous [*autonom*], like an arithmetic calculation or a geometrical construction.

Der Sinn der Bemerkung, daß die Arithmetik eine Art Geometrie sei, ist eben, daß die arithmetischen Konstruktionen autonom sind, wie die geometrischen, und daher sozusagen ihre Anwendbarkeit selbst garantieren.

Denn auch von der Geometrie muß man sagen können, sie sei ihre eigene Anwendung. [PG §15 pp. 600, 602 Cf. PR §111 p. 112]

The point of the remark that arithmetic is a kind of geometry is simply that arithmetical constructions are autonomous like geometrical ones and hence, so to speak, themselves guarantee their applicability.

For it must be possible to say of geometry too that it is its own application. [PG §15 pp. 306, 307 Cf. PR §111 p.132]

This difference between grammatical statements like ‘In English, the first word of every sentence is capitalized’ and mathematical calculations is critical. Mathematical calculations are autonomous, while sentences that express grammatical rules of their own language are

not. If the aforementioned sentence did not follow the rules of English, it would not make sense. However, the grammatical correctness of the statement does not guarantee that the rule expressed is an actual rule of English. By contrast, calculations cannot be grammatically correct unless they are rules of their calculus. The aforementioned English sentence and its negation are both grammatically correct. However, only one of them expresses a grammatical rule of English. By contrast, every correct mathematical calculation is a rule of the calculus. For a calculation, being correct, obeying the rules of the calculus and being a rule are the same.

Calculations are autonomous, because they do not express mathematical rules. They are the rules themselves. Grammatical rules are different from the English sentences that express them. Grammatical rules of language are not autonomous, while calculations are.

Die arithmetik hat es mit dem Schema $|||$ zu tun. – Aber redet denn die Arithmetik von Strichen, die ich mit Bleistift auf papier mache? – Die Arithmetik redet nicht von den Strichen, sie *operiert* mit ihnen. [PG §19 p. 654]

What arithmetic is concerned with is the schema $|||$. – But does arithmetic talk about the lines I draw with pencil on paper? – Arithmetic doesn't talk about the lines, it *operates* with them. [PG §19 p. 333]

The arithmetical calculation in Wittgenstein's example of §15 (see above display) is autonomous, because it is not about strokes or their division. It is a division itself. Dividing the strokes into groups of three is performing the calculation. The calculation is not *about* the division. The calculation *is* the division. It involves applying the rules for writing strikes and dividing them. The calculation is both one of the arithmetical rules for division and an application of them. Divisions are rules of division. Arithmetical calculations are rules of arithmetic. Calculations are rules of the calculus. Such is the autonomy of arithmetic.

B. *Anwendung* is an Essential Feature of Mathematics

Wenn man sagt: "es muß der Mathematik wesentlich sein, daß sie angewandt werden kann", so meint man, daß diese *Anwendbarkeit* nicht die eines Stückes Holz ist, von dem ich sage "das werde ich zu dem und dem anwenden können."

PG §17 p. 626

If we say "it must be essential to mathematics that it can be applied" we mean that its *applicability* isn't the kind of thing I mean of a piece of wood when I say "I will be able to apply it to this and that."

PG §17 p. 319

Logicians confuse two different kinds of grammatical application: external and internal. The external application gives rules for embedding the grammar of one language into that of another, quite different language. The internal application lives within the language. In either case, grammatical rules apply to propositions. As the name suggests, mathematical rules apply internally to propositions inside the calculus, and externally to propositions outside the calculus. However, external application is not part of the calculus itself. It lies entirely outside the calculus. Neither sort of application is mathematical. The only mathematical part of applied mathematics is the calculus.

Die Grammatik ist für uns ein reiner Kalkül. (Nicht die Anwendung eines auf die Realität.) [PG §15 p. 612]

Grammar is for us pure calculus (not the application of calculus to reality). [PG §15 p. 312]

From the perspective of the calculus, no significant difference exists between both applications. The rules of the calculus apply equally to propositions inside and outside the calculus. However, the external application connects the calculus with reality. The external application allows the use of calculations to solve non-mathematical problems. External application is an essential element of mathematics. It distinguishes mathematics from games.

The rest of mathematics is pure calculation [*Berechnung*]. In *Philosophical Grammar* §11,¹³ Wittgenstein writes the following.

Wenn ich in unserem Spiel 21×8 ausrechne, und wenn ich es tue, um damit eine praktische Aufgabe zu lösen, so ist jedenfalls die Handlung der Rechnung in beiden Fällen die Gleiche (und auch für Ungleichungen könnte in einem Spiele Platz geschaffen werden.) Dagegen ist mein übriges Verhalten zu der Rechnung jedenfalls in den zwei Fällen verschieden.

Die Frage ist nun: kann man von dem Menschen, der im Spiel die Stellung " $21 \times 8 = 168$ " erhalten hat, sagen, er habe herausgefunden, daß $21 \times 18 = 168$ sei? Und was fehlt ihm dazu? Ich glaube, es fehlt nichts, es sei denn eine Anwendung der Rechnung.

Die Arithmetik ein Spiel zu nennen, ist ebenso falsch, wie das Schieben von Schachfiguren (den Schachregel gemäß) ein Spiel zu nennen; denn das kann auch eine Rechnung sein. [PG §11 p. 573]

When I work out 21×8 in our game the steps in the calculation, at least, are the same as when I do it in order to solve a practical problem (and we could make room in a game for inequations also). But my attitude to the sum in other respects differs in the two cases.

Now the question is: can we say of someone playing the game who reaches the position " $21 \times 8 = 168$ " that he has found that $21 \times 8 = 168$? What does he lack? I think the only thing missing is an application for the sum.

Calling arithmetic a game is no more or less wrong than calling moving chessmen according to chess rules a game; for that might be a calculation too. [PG §11 p. 292]

Using calculations to solve mathematical, as well as non-mathematical problems is essential to mathematics. A calculus without external application would not be mathematical. External application provides the calculus with significance [*Bedeutung*]. It gives it certain 'importance for life' [*Lebenswichtigkeit*].

Es ist den Leuten unmöglich, die Wichtigkeit einer tatsache, ihre Konsequenzen, ihre Anwendung, von ihr selbst zu unterscheiden; die Beschreibung einer Sache von der Beschreibung ihrer Wichtigkeit. [PG §11 p. 578]

People cannot separate the importance, the consequences, the application of a

¹³. The section's title is 'The comparison between mathematics and a game' [*Die Mathematik mit einem Spiel Vergleichen*].

fact from the fact itself; they can't separate the description of a fact from the description of its importance. [PG §11 p. 295]

While Wittgenstein finds the external application of the calculus essential to mathematics, he does not find it foundational. Application does not play a foundational role in the calculus. In the solution of non-mathematical problems, the calculation neither becomes empirical nor acquires some extra *reality* it lacked before. The external application of a calculus does not make it more *real*.

Die unrichtige Idee ist, daß die Anwendung eines Kalküls in der Grammatik der wirklichen Sprache, ihm eine Realität zuordnet, eine Wirklichkeit gibt, die er früher nicht hatte. [PG §15 p. 610]

What is incorrect is the idea that the application of a calculus in the grammar of real language correlates it to a reality or gives it a reality that it did not have before. [PG §15 p. 311]

Making preparations for the application of arithmetic or any other mathematical calculus does not make sense. Since arithmetic is its own internal application, if the calculus exists, it has at least one application in itself. Application takes care of itself. For Wittgenstein, the logicians' project of circumscribing the totality of possible external applications of arithmetic using mathematical tools is impossible. The only way of picking out all of the legitimate applications of arithmetics would be using the expression 'legitimate application of arithmetic'.

Man könnte sagen: Wozu die Anwendung der Arithmetik einschränken, sie sorgt für sich selbst. (Ich kann ein Messer herstellen ohne Rücksicht darauf, welche Klasse von Stoffen ich damit werde schneiden lassen; das wird sich dann schon zeigen.) [PG §15 p. 601]

You could say: why bother to limit the application of arithmetic, that takes care of itself. (I can make a knife without bothering about what kinds of materials I will have cut with it; that will show soon enough.) [PG §15 p. 306]

Just like making a knife without considering what kind of materials it will cut, it is possible to construct a calculus without considering what kind of non-mathematical problems it will

solve. The application is entirely external to the calculus. The calculus is independent of its application. A calculus without external application is no less a calculus than an externally applied one.

C. On Consistency [*Widerspruchsfreiheit*]

Ich habe eine Arbeit von Hilbert gelesen über die Widerspruchsfreiheit. Mir kommt vor, daß diese ganze Frage falsch gestellt ist. Ich möchte fragen: *Kann* denn die Mathematik überhaupt widerspruchsvoll sein? Ich möchte die Leute fragen: Ja, was tut ihr denn eigentlich?

PR App. II. p. 305

I've been reading a work by Hilbert on consistency. It strikes me that this whole question has been put wrongly. I should like to ask: *Can* mathematics be inconsistent at all? I should like to ask these people: Look, what are you really up to?

PR App. II. p. 318.

The German word for consistency is '*Widerspruchsfreiheit*', which literally means 'freedom of contradiction'. For Wittgenstein, the notion of '*Widerspruchsfreiheit*' confuses two independent notions: the quality of a calculus being free from contradictions, and the necessary conditions for applying a calculus. For Wittgenstein, the two clearly do not match. The applicability of a calculus does not require the absence of contradictions. Mathematicians' use of '*Widerspruchsfreiheit*' misleads by implying that a calculus needs to be free from contradiction in order to be applicable. Furthermore, neither of the two notions confused in the foundational role of *Widerspruchsfreiheit* is provable. Hilbert and Ramsey's demand for consistency proofs as part of the foundation of mathematics is twice mistaken. On the one hand, application requires no proof of consistency. Since calculation is its own application, the applicability of the calculus needs no further proof but its own existence.

Wie wuare es denn, wenn ich einen solchen Kalkül anwenden will? Hätte ich bei der Anwendung kein gutes Gewissen, solange ich nicht die Widerspruchsfreiheit bewiesen habe? Aber kann ich denn so fragen? Kann ich den Kalkül anwenden, so habe ich ihn eben angewendet; es gibt kein nachträgliches Korrigieren. Was ich kann, das kann ich. Ich kann die

Anwendung nicht dadurch ungeschehen machen, daß ich sage: eigentlich war das keine Anwendung. [PR Appendix II p. 319]

But suppose I want to apply such a calculus? Would I apply it with an uneasy conscience if I hadn't already proved [its consistency]? but how can I ask such a question? If I can apply a calculus, I have simply applied it; there's no subsequent correction. What I can do, I can do. I can't undo the application by saying: strictly speaking that wasn't an application.. [PR Appendix II p. 332]

Formal proof cannot demonstrate the applicability of a calculus. Applicability is not a syntactic feature of the calculus. As a calculation, no proof can establish anything about its calculus or formal system. It cannot determine if the calculus is applicable or not. Any attempts at proving the applicability of a calculus formally will fail.

On the other hand, if 'consistency' consisted of the absence of contradictions, it would not be provable either. First of all, it is impossible to formulate contradictory rules. Formulating a rule is performing it as calculation, which requires its application as rule. To formulate a calculation rule, it must be applicable. In consequence, contradictory formulas do not exist. No rule can contradict itself or another rule.

Warum dürfen sich Regeln nicht widersprechen? Weil es sonst keine Regeln wären. [PG §14 p. 598]

Why may not the rules contradict one another? Because otherwise they would not be rules. [PG §14 p. 305]

For Wittgenstein, if inconsistency is the existence of a proposition like ' $p \cdot \sim p$ ' or ' $2 \times 2 = 5$ ' among the rules of the calculus, it is not a 'great misfortune' [*großes Unglück*]. The existence of such a rule cannot 'harm' [*schaden*] the calculus. It cannot make it useless or inapplicable. The existence of the rule sufficiently guarantees its applicability.

Wie wäre es etwa, wenn man in der Arithmetik zu den üblichen Axiomen die Gleichung $2 \times 2 = 5$ hinzunehmen wollte? Das heiße natürlich, daß das Gleichzeichen nun seine Bedeutung gewechselt hätte, d. h. Daß nun andere Regeln für das Gleichzeitige gälten. [PG §14 p. 595]

Suppose someone wanted to add the usual axioms of arithmetic the equation $2 \times 2 = 5$. Of course that would mean that the sign of equality had changed

its meaning, I. e. That there would now be different rules for the equal sign.
[PG §14 p. 303]

Using calculation to decide on a philosophical problem is the common mistake made in attempts to found mathematics on consistency proofs. In this section of the *Philosophical Grammar*, Wittgenstein clarifies a confusion in the philosophy of mathematics by separating philosophy from mathematics, and “putting each one in its place.” Wittgenstein does not deny the formal results of his adversaries. He challenges their philosophical *prose*. In this section in particular, he reclaims the notion of *Anwendung* for philosophy. He separates the philosophical problem of application from the formal concerns of consistency, interpretation, etc. Furthermore, he fully divorces the calculus itself from its application. This separation allows him to explain the joint autonomy and applicability of mathematical calculations. It explains how mathematical calculations are about nothing and still solve practical problems. It also explains how mathematical calculations can be both rules of mathematical calculus and syntactical rules of natural language.

IV. Conclusion

Wittgenstein bases his philosophy of mathematics during the thirties on the strong importance of context. For Wittgenstein, understanding the meaning of a proposition requires an analysis of its role in a larger system of propositions or in other sentences. Accordingly, the philosophy of mathematics must start by analyzing the contexts in which mathematical propositions are used.¹⁴ For Wittgenstein, as well as Ramsey, a philosophical analysis of numbers must contain an understanding of both their occurrences in purely mathematical contexts and in non-mathematical ones. In particular, understanding the meaning of

¹⁴. The importance of propositions' context of use will become more evident in Wittgenstein's later work. However, Wittgenstein's appreciation of the importance of mathematical calculi already shows recognition of the importance of context for understanding propositions.

numerical statements [*Zahlangaben*] requires an analysis of their roles in the contexts of their use. In the case of mathematical *Zahlangaben*, it requires understanding their role in calculation. In the case of non-mathematical ones, it requires an analysis of their role in the application of mathematics. Both analyses are essential for the full understanding of numbers and mathematics.

This chapter is an analysis of mathematical propositions and calculations in the context of their application. The grammatical nature of mathematical propositions manifests itself in their application. In Pt. II, section III of *Philosophical Grammar*, Wittgenstein offers an account of mathematical application where mathematical propositions are rules of grammar. They provide the rules for the creation and transformation of sentences, either in the calculus (*internal* application) or outside it (*external* application).

The *Anwendung* process starts with calculation. If a calculation is performed in accordance with the rules of the calculus, it is correct. Every calculation leaves behind a trace. If the calculation is correct, its trace is a (true) mathematical proposition. That proposition is its internal *Anwendung*. If the calculation is correct, the proposition is grammatically correct and, in consequence, a rule of its calculus. Applying this calculation externally requires embedding the calculus rules in another grammar, like natural English grammar. Mathematical propositions provide the grammar of English sentences. If the calculation is correct, the sentences are grammatically correct and express genuine propositions. The calculation guarantees that what they say is possible, but it does not justify them or guarantee their truth. If the calculation is incorrect, the expressions are nonsense. Next chapter gives a formal and detailed account of how this process takes place.

Chapter 5

Grammar

I. Introduction

In the thirties, grammar became a central issue in Wittgenstein's philosophy.¹

Wittgenstein's remarks about grammar from this period are some of most controversial.

For example, he wrote that the grammar of some signs completely determines their meaning,

Zur Grammatik gehört nicht, daß dieser Erfahrungssatz wahr, jener falsch ist. Zu ihr gehören alle Bedingungen (die Methode) des Vergleichs des Satzes mit der Wirklichkeit. Das heißt, alle Bedingungen des Verständnisses (des Sinnes). [PG §45, p. 168]

What belongs to grammar are the conditions (the method) necessary for comparing the proposition with reality. That is all the conditions necessary for the understanding (of the sense).[PG §45, p. 133]

Wittgenstein also maintained that investigations into the essence of things are grammatical investigations.² Most philosophers do not think that Wittgenstein's notion of grammar is the one in common use. The controversial nature of these statements begins with Wittgenstein's notion of grammar. The absence of an explicit definition in his published writings makes it difficult to justify his use of the word 'grammar.' Wittgenstein's brief explanation in the *Big Typescript* lacks specificity. The following pages develop a formal definition of grammar provisionally fitting the purposes of this investigation: (i) to compare Wittgenstein's notion of grammar with conventional grammars and determine whether Wittgenstein's use of 'grammar' is justified or not, (ii) to demonstrate that a grammatical analysis of Wittgenstein's kind can yield mathematical results, (iii) to allow for a more

¹. Grammar will remain central to Wittgenstein's philosophy beyond the middle period. In the *Philosophical Investigations*, he wrote: "Essence is expressed by grammar" and "Grammar tells what kind of object anything is." PI §371, 373

². BT 9, 38

precise definition of the grammatical nature of mathematics. This chapter pursues the first goal, while the following two chapters develop the others.

The rest of this chapter compares the analytical capacities of Wittgenstein's grammar and a conventional ones. The first section defines 'language'. The second section formally models the conventional notion of grammar, using basic mathematical and logical tools and the syntax of propositional calculus and English grammar as examples. The third section formalizes Wittgenstein's explicit thoughts about grammar during this period. Finally, the last section compares the analytic capacities of both grammatical notions. It compares their grammatical categories and equivalence relations. The comparison answers two questions, (i)? and (ii) Does Wittgenstein's approach make finer distinctions than conventional grammar? If it is possible to construct a conventional grammar out of Wittgenstein's categories, Wittgenstein's notion dovetails with conventional ones. If Wittgenstein's approach make finer distinctions than conventional grammar, Wittgenstein's grammar refines the conventional one. Answering these questions will establish if Wittgenstein's notion of grammar covers the same cases than any of the more familiar notions. Their answers might also explain why Wittgenstein created his own approach instead of using a conventional one.

The introduction of these two approaches employs an abstract, rule-based notion of grammar. It models grammar as a formal theory. Grammatical theories are special cases of formal theories. Grammatical theories are first order theories with a concatenation operator and several predicates: one for each grammatical category. The domain of the theory is the set of language expressions, and every proposition is of the form $x_1, x_2, \dots, x_n (C_1 x_1 \& C_2 x_2 \& \dots C_n x_n) \quad C_k(C(x_1, x_2, \dots, x_n))$ where C is a concatenation operator. Logical

notions such as satisfaction, truth, model, consistency, completeness, etc. have immediate application.

This formal reconstruction and analysis will ultimately shed light on Wittgenstein's philosophy of mathematics. Even though some of its formal results might well have importance on their own, logic is only a tool for the following philosophical analysis. Accordingly, an intuitive introduction precedes the introduction of every formal element. It assists readers in understanding the issues raised and interpreting the results.³

II. A Formal Background for the Discussion of Wittgenstein's Grammar

A. Language

Definition 1.1 [language]: Define a language L as the structure $\langle \Sigma, E, W \rangle$, where Σ is the *alphabet* or the finite, non-empty set of *words*, W and E are sets of finite strings of words, such that $(\Sigma \cup W) \cap E = \emptyset$ and every member of E is a substring of some member of W .

³. This formal approach to Wittgenstein's grammar is not the first. It is also not the first time that the formalization of Wittgenstein's notion of grammar compares it with linguist's grammar. In 1974, the Research Center for the Language Sciences of Indiana University published, as part of its 'Approach to Semiotics' paperback series, a very interesting book by Cecil H. Brown entitled *Wittgensteinian Linguistics* (The Hague: Mouton, 1974). Brown presented the contemporary linguistic controversy between pure and descriptive semiotics as a dispute between Chomsky's and Wittgenstein's views of language. Brown explicitly recognizes the evolution of Wittgenstein's philosophy of language. When talking about Wittgenstein's views on language, Brown refers to what he calls Wittgenstein's "ordinary language philosophy" (p.13): Wittgenstein's views after 1929 when "after having ignored the philosophy of language for some time, he took it up again." (p.15) "Readers who have encountered the works of both Chomsky and Wittgenstein are no doubt aware of the pronounced difference in the manner in which each explains the essential nature of patterned communication in the modality of natural language. This difference emerges at the most general levels of analysis. Chomsky is concerned with pure semiotics, the development of a language to talk about signs. Wittgenstein emphasizes descriptive semiotics, the study of actual sign use." (p. 13) In Brown's interpretation, Wittgenstein claims that "any language, be it artificial or natural, is understood not in terms of some other language, but in terms of itself, in the manner in which its signs are ordinarily used" (p. 17). Grammatical rules do not hide themselves. They are immediately identifiable in the surface structure of language (p. 90). By contrast, linguistic grammarians – at least of the most common Chomskian sort – locate grammar in the not-so-accessible deep structure of language. For Brown, "the deep structure of language is comparable to the logical systems or artificial languages of logical positivism. The deep structure is a kind of ideal language with which sentences of natural languages can be compared and consequently understood." Except for its pragmatic stress, Brown's formal treatment is very similar to the one this chapter presents.

W is the set of *acceptable* or *well-formed strings*, and E is the set of *expressions*. Every meaningful element of language is an expression.⁴ For example, in ordinary English, contains words like ‘apple’, ‘be’, ‘caring’, etc., E contains words like ‘dog’ and ‘caring’ and complex expressions like ‘my dog’ or ‘the yellow pencil on my desk.’ Finally, W contains all the grammatically correct sentences in English: ‘Try to remember my name’, ‘This is not the end of the line’, ‘Could you come here for a second?’ etc.

B. What is Grammar?

Grammar . . . is felt to be a term with a far wider meaning than that which a considered definition would propose or an elementary text illustrate. . . It is perhaps the vaguest term in the schoolmaster’s, if not the scholar’s vocabulary.

Ian Michael⁵

For most Wittgenstein’s scholars, ‘grammar’ in Wittgenstein has a “. . . meaning far wider than the ordinary one.”⁶ In the words of Hans-Johann Glock, Wittgenstein’s notion of grammar diverges from ordinary usage only in extension, not in sense.⁷ As evidence, Newton Garver quotes one of Wittgenstein’s letters to Moore, where he writes to be “. . . using the words ‘grammar’ and ‘grammatical’ in their ordinary sense but making them apply to things they do not ordinarily apply to.”⁸ Calling Wittgenstein’s use of the term ‘grammar’ “liberal”, Glock recognizes no significant difference between the ordinary

4. Wittgenstein calls expressions *words*.

5. Ian Michael, “Grammar, Divisions of Grammar and Parts of Discourse” in *English Grammatical Categories and the Tradition to 1800* (Oxford: Cambridge University Press, 1970) 37.

6. Finch, Henry LeRoy, *Wittgenstein: The Later Philosophy* (Atlantic Highlands: Humanities Press, 1997) p.149.

7. Glock, H. G., *A Wittgenstein Dictionary* (Cambridge: Blackwell, 1996) 152.

8. Garver, N., “Philosophy as Grammar” in *This Complicated Form of Life* (Chicago: Open Court, 1994) 150.

sense of grammar and Wittgenstein's. The meaning of 'grammar' covers both Wittgenstein's peculiar use and the ordinary one.⁹ In consequence, understanding Wittgenstein's peculiar assessment of grammar requires an investigation into the meaning of the word 'grammar.'

The German word 'Grammatik' – just like the English word 'grammar' – descends from the Greek ' γμμα ' meaning 'letter.' In classical Greek the expression γμμα () had two principal meanings. It addressed the phonetic (accentuation and pronunciation) and metaphysical values of letters. It also referred to the knowledge required to read and write.¹⁰ At the time 'grammar' entered the Latin language, its sense had gradually extended to include the general study of literature and language. In medieval usage, 'grammar' referred only to Latin grammar. In the seventeenth century it took on a more general meaning addressing language proficiency in Latin, English, French, etc¹¹. Nevertheless, the notion of 'universal grammar' – not the grammatical features of a particular language, but those common to all linguistic usage – did not appear until the work of Port Royal grammarians in the 18th Century. Even though it disappeared again in the middle of the nineteenth century, the work of Noah Chomsky launched a resurgence of universal grammar in the twentieth century.¹²

⁹. However, few authors venture a detailed characterization of grammar. For LeRoy Finch, for example, grammar is language and the phenomena connected with it in terms of its possibilities. Grammar lays down the limits of sense in language. It draws the line that separates sense from non-sense, expressibility from inexpressibility. Because it helps make sense of the evolution of Wittgenstein's philosophy, LeRoy Finch is not the only scholar to favor this interpretation. In Wittgenstein's middle period, grammar plays a similar role that logic did in his earlier work. During those years, Wittgenstein came to believe that logic was not the philosophical panacea he had mistaken it to be. Instead, logic constitutes a significant part, but not the whole of a larger grammatical philosophy. This dissertation's definition does not diverge far from LeRoy Finch's.

¹⁰. (Michael 1970, 24)

¹¹. Jackson, Howard: *Discovering Grammar* (Oxford: Pergamon Institute of English, 1985) 1.

¹². Serbat, Guy, *Casos y Funciones* (Madrid: Editorial Gredos, 1988) 74-84.

Today, the term ‘grammar’ has two uses. On the one hand, it refers to the structural features of a language. For example, the ‘Grammar of English’ refers to its structural features, instead of its semantics or pragmatics. On the other hand, ‘grammar’ also refers to the science or art describing (or prescribing) language’s structural traits. One talks about Chomskian or transformational grammars in this sense. Capitalizing the word ‘grammar’ in the first sense avoids confusion. Some authors prefer to mark the difference by calling ‘grammar’ the first one and ‘a grammar’ the second.

These two meanings of grammar have competed with each other since the seventeenth century. For descriptive grammarians, grammar is a science, a study of a set of phenomena. For prescriptive grammarians, it is an art: the skill or technique of using the language well. Ben Jonson, George Kittredge and L. Murray are well-known prescriptive grammarians. In contrast, Francis Bacon was a descriptive grammarian. Today, most consider the prescriptive and descriptive aspects of grammar inseparable. In the introductory pages of his *Discovering Grammar*, Howard Jackson writes,

In the event, although different basic attitudes prevail, the distinction is probably not so clear cut as the terms ‘descriptive’ and ‘prescriptive’ imply. To be sure, prescriptive grammarians included rules in their grammars, such as “you should not end a sentence with a preposition”; but in so doing they still had to describe what a ‘sentence’ and a ‘preposition’ are. And a descriptive linguist producing a grammar of modern English, for example, has to make a choice of which English usage he is going to describe; and he would usually select the ‘standard’ variety, perhaps even ‘standard educated usage’, and by so doing he would have indulged in an implicit prescription.¹³

Nevertheless, the descriptive/prescriptive dichotomy survives in the current opposition between *school* and *linguistic* grammar. School grammarians stress the prescriptive aspect

¹³. (Jackson 1985, 2)

of grammar, while linguistic grammarians emphasize the descriptive dimensions of their science.¹⁴

Sometimes ‘grammar’ refers to the basic structural aspects of a language. Other times, it means only the ‘correct’ or ‘standard’ usage of the language. This makes specifying both the aspect of the language and the kind of grammar referred to vital.

Meaning of Grammar

	Linguist Grammar	School Grammar
Aspect of Language	General Usage	Correct or Standard Usage
Study of Language	Descriptive	Prescriptive

For the purposes of this dissertation, ‘grammar’ describes the structural aspects of language in its general use. For further clarification, it restricts ‘grammar’ to the syntactic structure of sentences. Grammar consists of two sub-components: morphology and syntax. Morphology deals with the form of words, while syntax deals with meaningful word combinations.¹⁵ ‘Syntax’ has its roots in the Greek word for ‘arrangement’. It addresses the possible arrangements, patterns or orders of words as well as the differences in meaning that the various orderings bring out.¹⁶

14. In *Traditional Grammar* Jewell A. Friend argues against the identification of the prescriptive tradition with schoolroom grammar. At the end of his book’s introduction, he lists seven points of divergence. Amongst them, schoolroom grammar does not distinguish between written and oral forms of language, also ignoring the distinction between lexical and grammatical meaning. (Carbondale: Southern Illinois University Press, 1976) i - xi.

15. Huddleston, Rodney, *English Grammar: An Outline* (Cambridge: Cambridge University Press, 1988) 1.

16. (Jackson 1985, 3)

C. The Conventional Approach

To define the ordinary sense of ‘grammar’, this section synthesizes the essential features of sophisticated linguistic grammars like Chomsky’s and everyday school grammar. All conventional grammars distribute the expressions of the language into several categories, providing explicit rules for combining these expressions in a way that acknowledges the grammatical categories to which they belong. All conventional grammars present this basic feature.

Reduced to the simplest possible terms, the methods of structural grammarians consist of breaking the flow of spoken language into the smallest possible units, sorting them out, and studying the various ways in which these units are joined in meaningful combinations.¹⁷

The conventional presentation of syntax for the predicate calculus exemplifies this feature. The basic symbols – broken into categories, and a recursive definition for terms and well-formed formulas – determine the language of predicate calculus. School grammar has a similar presentation.

Most traditional “school” grammars begin by defining and classifying . . . words into part-of-speech categories, and proceed from there to more inclusive sentence components until they arrive at a discussion of the sentence itself.¹⁸

The first step in the process of learning the grammar of a language is learning the vocabulary and the grammatical categories. Learning that ‘duck’ is a noun and ‘she’ a pronoun is not sufficient. To learn that ‘duck’ is a singular common noun and that ‘she’ is a singular, feminine, third person, personal pronoun is also necessary. The categories to which an expression belongs exhaust its grammar. The next step is to learn which sequential combinations of categories are grammatically correct and which are not. Determining which

¹⁷. Jeanne H. Herndon, *A Survey of Modern Grammars* (New York: Molt, Rinehart & Winston, 1970) 65.

¹⁸. Joseph La Palombara, *An Introduction to Grammar* (Cambridge: Winthrop, 1976) 23.

expression sequences are meaningful requires knowing to which categories words belong and which categories combine into grammatically acceptable expressions.

A set \mathbf{C} of grammatical categories, an interpretation function I and a set \mathbf{S} of word combination rules constitute an abstract grammar $\mathbf{G} = \langle \mathbf{C}, I, \mathbf{S} \rangle$. Besides grammatical categories like ‘noun’ in school grammar, or ‘statement letter’ in the syntax of propositional logic, a grammar also includes an interpretation function that determines which words of the language belong to which categories. This function maps the category ‘noun’ to the set of nouns in the language. Finally, the set of rules \mathbf{S} sets parameters for the combination of expressions into other expressions. For example, consider SEN as the category ‘sentence’ and CON as the category ‘conjunction’. The rule SEN CON SEN SEN says that the concatenation of a sentence, a conjunction and a sentence creates another sentence.

Definition 1.2.1 [grammatical language]: Let C, C_0, C_1, C_2, \dots and the arrow \rightarrow be the *basic symbols* of grammatical language.

Definition 1.2.1.1 [categorical symbols]: C, C_0, C_1, C_2 are the *categorical symbols* of grammatical language.

Definition 1.2.1 [basic symbols]: Let C, C_0, C_1, C_2, \dots and the arrow \rightarrow be the *basic symbols* of grammatical language.

Definition 1.2.2 [grammatical formulae]: Every sequence of categorical symbols of the form $C_0 C_1 \dots C_n \rightarrow C$ is a *well-formed grammatical formula*.

Definition 1.2.2.1 [antecedent and resulting categorical symbols]: Given a grammatical formula $C_0 C_1 \dots C_n \rightarrow C$, the *antecedent* categorical symbols of the rule are $C_0 C_1 \dots C_n$, and C is the *resulting* categorical symbol.

Definition 1.2.2.2 [degree of a grammatical formula]: The *degree* a grammatical formula $C_0 C_1 \dots C_n$ is n , the number of antecedent categorical symbols.

Definition 1.2 [grammatical theory]: Given a set of categories \mathbf{C} , a grammatical theory \mathbf{S} is a set of well-formed expressions of the language — called the *rules* of the grammar— such that, for all $C \in \mathbf{C}$, C occurs in some rule $s \in \mathbf{S}$.

Definition 1.2.3 [interpretation]: Given a language $\mathbf{L} = \langle \mathbf{C}, \mathbf{E}, \mathbf{W} \rangle$ and a set of categorical symbols \mathbf{C} , an *interpretation* I is a function from \mathbf{C} into the power set of the expressions, $I: \mathbf{C} \rightarrow \mathcal{P}(\mathbf{E})$, such that $I[C] = E$.

Definition 1.2.4 [application]: A rule $C_1 C_2 \dots C_n \rightarrow C$ *applies* to an n -tuple of expressions $\langle e_1, e_2, \dots, e_n \rangle$ iff, for all $1 \leq i \leq n$, $e_i \in I(C_i)$.

Definition 1.2.5 [result of an application]: A concatenation of expressions $e_1 \{e_2 \{ \dots \{e_n$ is *the result of applying* a rule $C_1 C_2 \dots C_n \rightarrow C$ to an n -tuple of expressions $\langle e_1, e_2, \dots, e_n \rangle$ iff the rule applies to the n -tuple.

Definition 1.2.6 [satisfaction]: A sequence of expressions $\langle e_1, e_2, \dots, e_n \rangle$ *satisfies* a rule $C_1 C_2 \dots C_n \rightarrow C$ iff, if the rule applies to the n -tuple $\langle e_1, e_2, \dots, e_n \rangle$, then $e_1 \{e_2 \{ \dots \{e_n \in I(C)$ where $e_1 \{e_2 \{ \dots \{e_n$ is the result of applying $C_1 C_2 \dots C_n \rightarrow C$ to $\langle e_1, e_2, \dots, e_n \rangle$.

Definition 1.2.7 [grammatical truth]: A rule s of degree n is *true* for a given interpretation I , written $\vdash_I s$, iff every n -tuple of language expressions satisfies s .

Definition 1.2.8 [model of a theory]: An interpretation I *models* a grammatical theory \mathbf{S} if every rule in \mathbf{S} is true for I .

Definition 1.2.9 [consistency]: A grammatical theory S is *consistent*, if some interpretation function I models S .

Example 1.2.10: Consider the grammatical theory S with categories SENT, NOUN and VERB and the single rule s : NOUN VERB SENT. Interpretation I assigns to NOUN the set {Bill} and to VERB {runs}. If I assigns any set of expressions including 'Bill runs' to SENT, then s is true for I and I models S . However, if another interpretation J agrees with I on NOUN and VERB but assigns to SENT a set of expressions not including 'Bill runs', then s is not true for J and J fails to model S .

Note 1.2.11: Those familiar with the conventions of Chomskian or generative grammar will recognize that the above notion of abstract grammar is divorced from all questions of computability. It sets no *a priori* limit on the number of rules that may enter into a grammatical theory, except that they cannot comprise a proper class. Indeed, as will appear later, any language L whose well-formed expressions make up a set will have a grammar in this sense. The above notion of abstract grammar is 'purely logical', yielding a form of decompositional description of a language failing to constrain a finite machine's ability to recognize or decide appropriate sequences.

Definition 1.3 [conventional equivalence]: Given a set of categories C and an interpretation I , define the relation of *conventional equivalence* \sim on E^2 by:

$$e_1 \sim e_2 \text{ iff } C \in C \text{ [} (e_1 \in I(C)) \text{ and } (e_2 \in I(C)) \text{].}$$

Two expressions are conventionally equivalent if they belong to exactly the same categories.

Definition 1.4 [decomposition]: Let S be a grammatical theory. An expression e *decomposes* into a set of expressions B iff (1) a rule s in S applies to the n -tuple of expressions $\langle e_1, e_2, \dots, e_n \rangle$, (2) an expression e_i occurs in the n -tuple $\langle e_1, e_2, \dots, e_n \rangle$ if and

only if it belongs to B , and (2) the application of rule s to n -tuple $\langle e_1, e_2, \dots, e_n \rangle$ results in expression e .

An expression decomposes into a set of other expressions if a rule in the grammatical theory explains how to combine those expressions into the original one. Given that other expressions might exist beyond the basic symbols and acceptable strings (E might be larger than W), other grammars might decompose a string in different ways. Consider the expression ‘My dog is dead.’ If a grammatical rule said that combining a singular nominal expression (like ‘my dog’) with the singular third person indicative present form of the verb to be (‘is’) and an adjective (like ‘dead’) resulted in a sentence, then ‘My dog is dead’ would decompose into the three expressions ‘my dog’, ‘is’ and ‘dead’. On the other hand, if another rule stated that subjects combined with predicates matching in number form sentences, then ‘My dog is dead’ would decompose into only two expressions, ‘My dog’ and ‘is dead’. In consequence, the set of expressions into which an expression decomposes depends upon the rules in the grammatical theory.

Definition 1.5 [construction as, code]: Given a grammatical theory S , a set of categories C and an interpretation function I , a *construction* of an expression e as member of category C is a sequence $P = \langle p_1, p_2, \dots, p_n \rangle$ of expressions – said to *occur in* P , such that there is a sequence of categories $\langle C_1, C_2, \dots, C_n \rangle$ —called the *code* of P , where for all $1 \leq i \leq n$,

1. $C_i \in C$
2. $p_i \in I(C_i)$
3. If p_i occurs in P , then applying a rule $s = D_1 D_2 \dots D_k \in C_i$ in S to a k -tuple of expressions $\langle e_1, e_2, \dots, e_k \rangle$, such that for all $1 \leq j \leq k$, $D_j = C_{g(j)}$ and $e_j \in I(C_{g(j)})$, results in p_i . In that case, p_i occurs in P in virtue of s

4. If p_i , then $I(C_i)$

5. $p_n = e$, and

6. $C_n = C$.

Proposition 1.6: Every expression occurring in a construction has a construction itself. \sim

Definition 1.7 [proof]: If $e \in W$ and $I(C)=W$, then the construction of e as C is a *proof*.

Definition 1.8 [tree]: Given a construction $P = \langle p_1, p_2, \dots, p_n \rangle$ of code $\langle C_1, C_2, \dots, C_n \rangle$

for an acceptable string $w \in W$ in a grammatical theory \mathbf{S} , a set of categories \mathbf{C} and an interpretation function I , a *tree* of P is a labeled directed graph $\langle \mathbf{T}, \langle \rangle \rangle$ such that:

1. For all $p \in P$, $p = \text{label}(t)$ for some node $t \in \mathbf{T}$
2. If $t_1 < t_k, t_2 < t_k, \dots, t_{k-1} < t_k$, then $s = D_1, D_2 \dots D_{k-1}$ D_k is a rule in \mathbf{S} such that
for all $1 \leq i \leq k$, $\text{label}(t_i) = p_j$ and $D_i = C_j$ for some $1 \leq j \leq n$
3. For all $t \in \mathbf{T}$, $\text{label}(t) = w$, iff $t < u$ for some $u \in \mathbf{T}$
4. For all $t_1, t_2, t_3 \in \mathbf{T}$, if $t_1 < t_2$ and $t_1 < t_3$, then $t_2 = t_3$
5. For all $t \in \mathbf{T}$, $\text{label}(t) = w$ iff $u < t$ for some $u \in \mathbf{T}$

Definition 1.9 [occurrence of an expression]: Let \mathbf{T} be the tree of a construction P , then an expression *occurs* in \mathbf{T} iff it occurs in P .

Definition 1.10 [correspondance]: A proof $P = \langle p_1, p_2, \dots, p_n \rangle$ *corresponds* to a tree \mathbf{T} iff for all nodes $t_1 < t_2$ in \mathbf{T} , there are $1 \leq i \leq j \leq n$ such that $e_i = \text{label}(t_1)$ and $e_j = \text{label}(t_2)$.

Definition 1.11 [theorem]: An expression w is a *theorem* of a grammatical theory \mathbf{S} , written \mathbf{S} / w , iff w has a proof in \mathbf{S} .

Definition 1.12 [completeness]: A grammatical theory \mathbf{S} , together with a set of categories \mathbf{C} and an interpretation \mathbf{I} , is *complete* for a language \mathbf{L} if all acceptable strings of \mathbf{L} are theorems of \mathbf{S} and conversely. In other words, \mathbf{S} is complete whenever $w \in W$ iff $\vdash w$.

Definition 1.13 [conventional grammar]: Given a language \mathbf{L} , a *conventional grammar* for \mathbf{L} is a triple $\langle \mathbf{S}, \mathbf{I}, \mathbf{C} \rangle$ where \mathbf{I} models \mathbf{S} , and \mathbf{S} , together with \mathbf{C} and \mathbf{I} , is complete for \mathbf{L} .

D. Example: The Language of Propositional Calculus

Any conventional presentation of propositional calculus syntax fits the previous definition of a conventional grammar. Take, for example, Elliot Mendelson's presentation of propositional calculus in the third edition of his *Introduction to Mathematical Logic*.¹⁹ Displaying the syntax of propositional calculus as a language in the aforementioned form $\langle \Sigma, E, W \rangle$ and reconstructing Mendelson's recursive definition of well-formed formula as a conventional grammar $\langle \mathbf{S}, \mathbf{I}, \mathbf{C} \rangle$ is easy. Nevertheless, besides showing that the grammar corresponds to the syntax, it also illustrates the concepts defining the notion of conventional grammar.

1. The Language of Propositional Calculus

The Language of Propositional Calculus \mathbf{L} is the structure $\langle \Sigma, E, W \rangle$ where Σ is the set of basic symbols $\{ \neg, \wedge, \vee, \rightarrow, \leftrightarrow, (,), A_1, A_2, A_3, \dots \}$, W is the set of well-formed formulas of propositional calculus and E contains both the basic symbols and well-formed formulas, so that $E = \Sigma \cup W$.

2. The Grammar of Propositional Calculus

The following is Mendelson's definition of a well-formed formula [wff]:

¹⁹ Elliot Mendelson, *Introduction to Mathematical Logic* (Monterrey: Wadsworth & Brooks/Cole Advanced Books & Software, 1987)

1. The vocabulary of language L contains \neg , \wedge , \vee , \rightarrow , \leftrightarrow , and the letters A_i with positive integers i as subscripts: A_1, A_2, A_3, \dots . The symbols \neg and $\wedge, \vee, \rightarrow, \leftrightarrow$ are *primitive connectives*, and the letters A_i are *statement letters*.
2. (a) All statement letters are well-formed formulas (wfs).
 (b) If A and B are wfs, so are $(\neg A)$ and $(A \rightarrow B)$.

An expression is a wf only if it can be shown to be a wf on the basis of clauses (a) and (b).

It is possible to reconstruct this definition as a conventional grammar the following way.

$$\mathbf{P} = \langle \mathbf{S}, \mathbf{I}, \mathbf{C} \rangle$$

$$\mathbf{C} = \{ \text{neg}, \text{arr}, \text{lpar}, \text{rpar}, \text{sl}, \text{wff} \}$$

$$\mathbf{I}(\text{neg}) = \{ \neg \}$$

$$\mathbf{I}(\text{arr}) = \{ \rightarrow \}$$

$$\mathbf{I}(\text{lpar}) = \{ (\}$$

$$\mathbf{I}(\text{rpar}) = \{) \}$$

$$\mathbf{I}(\text{sl}) = \{ A_1, A_2, A_3, \dots \}$$

$$\mathbf{I}(\text{wff}) = \text{well-formed formulas}$$

The syntax of propositional calculus contains six grammatical categories: corner (*neg*), right arrow (*arr*), open parenthesis (*lpar*), closed parenthesis (*rpar*), statement letters (*sl*) and well-formed formulas (*wff*). Also, it contains three formation rules.

$$\mathbf{S} = \{ s_1, s_2, s_3 \}$$

$$s_1 = \text{sl} \quad \text{wff}$$

First, all statement letters are well-formed formulas. If a is an expression of category *sl* (a statement letter), then it is an expression of category *wff*, i.e. a well-formed formula.

$$s_2 = \text{lpar wff arr wff rpar} \quad \text{wff}$$

Second, if A and B are arbitrary wfs, so is $(A \rightarrow B)$. If (i) c and d are well-formed formulas – expressions of category wff – (ii) a is an expression of category $lpar$ (an open parenthesis), (iii) e is an expression of category $rpar$ (a closed parenthesis), and (iv) b is an expression of category arr (a right arrow), then the concatenation $a\{b\{c\{d\{e$ is a well-formed formula itself.

$$s_3 = lpar \ neg \ wff \ rpar \ \ wff$$

Third, if A is a wf, so is $(\neg A)$. If (i) d is an expressions of category wff (a well-formed formula), (ii) a is an expression of category $lpar$ (left parenthesis), (iii) c is an expression of category $rpar$ (right parenthesis), and (iv) b is an expression of category neg (corner), then the concatenation $a\{b\{c\{d,$ is an expression of category wff , that is, a well-formed formula itself.

Proposition 1.14: I models S .

Proof: Assume not. Then, a rule $s \in S$ is not true for I . Hence, a sequence of expressions $\langle e_1, e_2, \dots, e_n, \dots \rangle$ does not satisfy s . Since $S = \{s_1, s_2, s_3\}$, three cases must be considered.

Case 1. $s = s_1$. $\langle e_1, e_2, \dots, e_n, \dots \rangle$ does not satisfy $sl \rightarrow wff$. Hence, $sl \rightarrow wff$ applies to $\langle e_1 \rangle$, but the result of applying $sl \rightarrow wff$ to $\langle e_1 \rangle$ does not belong to $I(wff)$. Since $sl \rightarrow wff$ applies to $\langle e_1 \rangle$, $e_1 \in I(sl)$. $I(sl) = \{A_1, A_2, A_3, \dots, A_n\}$. Therefore, e_1 is a statement letter. In other words, $e_1 = A_i$ for some $1 \leq i \leq n$. Also, it is the result of applying $sl \rightarrow wff$ to $\langle e_1 \rangle$. In consequence, A_i does not belong to $I(wff)$. But $I(wff)$ is the set of well-formed formulas. This would mean that the statement letter A_i is not a well-formed formula, which is false.

Case 2. $s = s_2 \cdot \langle e_1, e_2, \dots, e_n, \dots \rangle$ does not satisfy $lpar\ wff\ arr\ wff\ rpar\ wff$. Hence, $lpar\ wff\ arr\ wff\ rpar\ wff$ applies to $\langle e_1, e_2, e_3, e_4, e_5 \rangle$, but the result of applying $lpar\ wff\ arr\ wff\ rpar\ wff$ to $\langle e_1, e_2, e_3, e_4, e_5 \rangle$ does not belong to $I(wff)$. Since $lpar\ wff\ arr\ wff\ rpar\ wff$ applies to $\langle e_1, e_2, e_3, e_4, e_5 \rangle$, $e_1 \in I(lpar)$, $e_2 \in I(wff)$, $e_3 \in I(arr)$, $e_4 \in I(wff)$ and $e_5 \in I(rpar)$. Therefore, e_1 is an open parenthesis, e_2 is a well-formed formula, e_3 is the right arrow, e_4 is also a well-formed formula and e_5 is the closed parenthesis. The result of applying $lpar\ wff\ arr\ wff\ rpar\ wff$ to $\langle e_1, e_2, e_3, e_4, e_5 \rangle$ is a sequence of the form $(A \rightarrow B)$ where A and B are wfs. Hence, it does not belong to $I(wff)$. But $I(wff)$ is the set of well-formed formulas. This would mean that $(A \rightarrow B)$ is not a well-formed formula, which is false.

Case 3. $s = s_3 \cdot \langle e_1, e_2, \dots, e_n, \dots \rangle$ does not satisfy $lpar\ neg\ wff\ rpar\ wff$. In consequence, $lpar\ neg\ wff\ rpar\ wff$ applies to $\langle e_1, e_2, e_3, e_4 \rangle$, but the result of $lpar\ neg\ wff\ rpar\ wff$ to $\langle e_1, e_2, e_3, e_4 \rangle$ does not belong to $I(wff)$. Since $lpar\ neg\ wff\ rpar\ wff$ applies to $\langle e_1, e_2, e_3, e_4 \rangle$, $e_1 \in I(lpar)$, $e_2 \in I(neg)$, $e_3 \in I(wff)$ and $e_4 \in I(rpar)$. Hence, e_1 is an open parenthesis, e_2 is the corner, e_3 a well-formed formula and e_4 is the closed parenthesis. The result of applying $lpar\ neg\ wff\ rpar\ wff$ to $\langle e_1, e_2, e_3, e_4 \rangle$ is a sequence of the form $(\neg A)$ where A is a wf. Thus, it does not belong to $I(wff)$. But $I(wff)$ is the set of well-formed formulas. This would mean that $(\neg A)$ is not a well-formed formula, which is false.

Therefore, I models S for the language of propositional calculus L . \sim

Proposition 1.15: All acceptable strings $w \in W$ have a proof.

Proof: Let $w \in W$.

Induction on the complexity of w .

Base: w is a statement letter.

w and $w \in I(sI)$. Hence, $\langle w \rangle$ is the construction of w with code $\langle sI \rangle$. Let w be a wf of complexity n .

Inductive hypothesis: all wfs in L of complexity $m < n$ have a construction in P .

Case 1. w is of the form $\neg A$.

Since A 's complexity is less than n , the inductive induction applies to it. There is a proof

$P_A = \langle e_1, e_2, e_3, \dots, e_n \rangle$ of A with code $C_A = \langle C_1, C_2, C_3, \dots, C_n \rangle$.

Claim: $P_w = \langle e_1, e_2, e_3, \dots, e_n, \neg, w \rangle$ is a proof of w with code $C_w = \langle C_1, C_2, C_3, \dots, C_n, neg, wff \rangle$.

Proof: Let e_i be an expression in P_w .

Case 1: $i \leq n$. $e_i \in P_A$. Hence, either e_i is a statement letter or e_i decomposes into some of the previous expressions in the sequence.

Case 2. $e_i = \neg$. Hence, e_i is a statement letter and e_i is *neg*.

Case 3. $e_i = w$. Since P_A is a proof of A , $e_n = A$. From s_3 , w decomposes into \neg and A , both occurring before w in P_w . Therefore, P_w is a proof of w .

Case 2. w is of the form $A \rightarrow B$.

Since A and B are of complexity less than n , the inductive induction applies to them. There

is a proof $P_A = \langle e_1, e_2, e_3, \dots, e_n \rangle$ of A with code $C_A = \langle C_1, C_2, C_3, \dots, C_n \rangle$ and a proof

$P_B = \langle e_{n+1}, e_{n+2}, e_{n+3}, \dots, e_{n+m} \rangle$ of B with code $C_B = \langle C_{n+1}, C_{n+2}, C_{n+3}, \dots, C_{n+m} \rangle$.

Claim: $P_w = \langle e_1, e_2, e_3, \dots, e_n, e_{n+1}, \dots, e_{n+m}, \rightarrow, w \rangle$ is a proof of w with code $C_w = \langle C_1,$

$C_2, C_3, \dots, C_n, C_{n+1}, \dots, C_{n+m}, arr, wff \rangle$.

Proof: Let e_i be an expression in P_w .

Case 1: $i \leq n$. $e_i \in P_A$. Hence, either e_i or e_i decomposes into some of the previous expressions in the sequence.

Case 2: $n < i \leq n+m$. $e_i \in P_B$. Hence, either e_i or e_i decomposes into some of the previous expressions in the sequence.

Case 3. $e_i = \neg$. Hence, e_i and e_i are *arr*.

Case 4. $e_i = w$. Since P_A is a proof of A , $e_n = A$. Since P_B is a proof of B , $e_{n+m} = B$. From s_2 , w decomposes into \neg , A and B , occurring before w in P_w . Therefore, P_w is a proof of w .

~

Proposition 1.16: The structure $\mathbf{P} = \langle \mathbf{S}, \mathbf{I}, \mathbf{C} \rangle$ is a conventional grammar.

Proof: Directly from the previous two propositions 1.14 and 1.15. ~

E. Strong, Redundant and Trivial Grammars

Section C presented the minimal requirements for an abstract grammar. This section explores two special kinds of abstract grammars: strong and trivial. A strong grammar contains constructions for every expression as every category it belongs to. The syntax of first order logic is a clear example of a strong grammar, because its only categories are ‘well-formed formula’ and ‘basic symbols’.

The grammatical categories of a trivial grammar are all singletons of expressions. As their names suggest, trivial and strong grammars represent the two extremes of grammatical expressibility. A strong grammar’s categorical distinctions are the finest possible, while those of a trivial grammar are the weakest. Any grammatical distinction not in a strong grammar is superfluous. Any grammar containing superfluous distinctions is redundant.

1. Strong Grammars

Definition 1.17 [strong grammar]: A conventional grammar $\langle \mathbf{S}, \mathbf{I}, \mathbf{C} \rangle$ is *strong* if, for every expression $e \in E$ and every category $C \in \mathbf{C}$, if $e \in I(C)$, then e has a construction as C .

Example 1.17.1: The above grammar \mathbf{P} of the syntax of propositional calculus is strong.

Proof: Let $e \in E$. Since $E=W$, either $e \in W$ or $e \in I$. In the first case, where $e \in W$, if $e \in C \in \mathbf{C}$, $\langle e \rangle$ constructs e as C with code $\langle C \rangle$. In the second case, where $e \in W$, from proposition 1.13, e has a proof P_e . Also, P_e constructs e as *wff*. Except for statement letters, which belong to S , no *wff* belongs to any other category except *wff*. This proves that every expression in \mathbf{L} has a construction in \mathbf{P} . In other words, \mathbf{P} is strong.

Proposition 1.18: For every language such that $E = W$, every conventional grammar is strong. ~

Recognizing which grammatical distinctions give rise to a language's significant syntactic features is critical. Imagine a grammar for the syntax of propositional calculus containing, instead of a category for statement letters $SLET = \{ A_i \mid i \in \mathbf{N} \}$, two categories $ODSL = \{ A_i \mid i \in \mathbf{N} \text{ and } i \text{ is odd} \}$ and $EVSL = \{ A_i \mid i \in \mathbf{N} \text{ and } i \text{ is even} \}$. It is easy to imagine how, instead of a rule saying all propositional symbols are well-formed formulas, the grammar had two such rules: one for each of the categories $ODSL$ and $EVSL$. To an extent, this distinction is superfluous, if compared with the distinction between the negation and open parenthesis symbols. Consider an English grammar which not only distinguishes between adjectives and adverbs but also between nouns which contain more than three occurrences of the letter 'e' and nouns which do not. Clearly, this latter distinction does not have the grammatical significance of the former. The grammar resulting from the collapse of

two categories with superfluous distinctions neither stops being a grammar nor loses its strength.

2. Redundant Grammars

Definition 1.19 [redundant grammar]: Let $\langle S, I, C \rangle$ be a conventional grammar for language L such that $A, B \in C$. Let AB be a categorial symbol not in C . Let C^* be $(C - \{A, B\}) \cup \{AB\}$. Let S^* be the grammatical theory resulting from substituting every occurrence of A or B by AB , and let I^* be the function such that for all $C \in C$, $I^*(C) = I(A \cup B)$ if $C = AB$, and $I^*(C) = I(C)$, otherwise. Grammar $\langle S, I, C \rangle$ is *redundant* if the following conditions hold: (i) $\langle S^*, I^*, C^* \rangle$ is also a conventional grammar for language L , and (ii) if $\langle S, I, C \rangle$ is strong, so is $\langle S^*, I^*, C^* \rangle$.

Example 1.19.1: The above grammar P for the syntax of propositional calculus is not redundant.

Proof: Proving the non-redundancy of P requires demonstrating that the collapse of any two categories in C affects the conventionality or strength of P . This seems false, because it is possible to take a well-formed-formula and substitute one of its component expressions for an expression not of the same category, to obtain a new-well-formed formula. This is possible by substituting molecular subformulas for atomic ones, like substituting ' A_3 ' for ' $(A_1 \cup A_2)$ ' in ' $\neg (A_1 \cup A_2)$ ' to obtain ' $\neg A_3$ ' (claim 1). This is the only collapse of categories that respects truth (claim 3). However, this substitution does not respect the completeness of P (claim 2). P^* is not complete for L , because it lacks proofs for every well-formed formula. In particular, it offers no proof for single letters being well-formed. Therefore, P^* is not a conventional grammar for L .

$slorwff$ is a categorial symbol not in C . Let C^* be $(C - \{sl, wff\}) \cup \{slorwff\}$. Let S^* be the grammatical theory resulting from substituting every occurrence of sl or wff by

$slorwff$, and let I^* be the function such that for all $C \in \mathbf{C}$, $I^*(C) = I(sl) \vee I(wff)$ if $C = slorwff$, and $I^*(C) = I(C)$, otherwise.

Claim 1. $\mathbf{P}^* = \langle \mathbf{S}^*, I^*, \mathbf{C}^* \rangle$ models the language of propositional calculus \mathbf{L} . In other words, every rule in \mathbf{S}^* is true for \mathbf{L} .

Proof: Every statement letter by itself is also a well-formed formula: $I(sl) \vee I(wff)$. Hence, $I(slorwff) = I(wff) \vee I(sl) = I(wff)$. Hence, the substitution of wff for $slorwff$ does not affect the interpretation of the rule. Similarly, every occurrence of sl or wff in \mathbf{S} could stand for $slorwff$ without affecting the truth of the theory. First, $s_1 = sl \vee wff$ is the only rule in which sl occurs. Hence, it is the only rule in \mathbf{S} that is significantly transformed in \mathbf{S}^* . Substituting $slorwff$ from s_1 for every occurrence of sl results in the rule $slorwff \vee slorwff$, but obviously, $\vdash_1 wff \vee wff$. Therefore, $\vdash_{I^*} slorwff \vee slorwff$. End of proof for claim 1.

Claim 2. The grammar $\mathbf{P}^* = \langle \mathbf{S}^*, I^*, \mathbf{C}^* \rangle$ resulting from collapsing the categories wff and sl into $slorwff$, is not complete.

Proof: No proof for statement letters as well-formed formulas would exist. Remember that this definition of proof includes the condition that every basic symbol in a proof must belong to a category whose interpretation includes only basic symbols. The syntax of propositional calculus satisfies this condition, precisely because a separate category exists for well-formed formulas which are also basic symbols – statement letters. In other words, $I(sl) \vee I(wff)$. However, without sl , this is no longer true. End of proof of claim 2.

Claim 3. Let C_1 or C_2 be the category resulting from collapsing different categories C_1 and C_2 in \mathbf{C} such that C_1 is neither *sl* nor *wff*, and let $\mathbf{P}^* = \langle \mathbf{S}^*, \mathbf{I}^*, \mathbf{C}^* \rangle$ be the grammar constructed according to definition 1.19. At least one rule s^* in \mathbf{S} is false for \mathbf{L} .

Proof: Case 1. C_2 is *sl*. Since $s_1 = sl \quad wff \quad \mathbf{S}$, $s_1^* = C_1 or sl \quad wff \quad \mathbf{S}^*$. Since C_1 is not *wff*, s^* is false for \mathbf{L} . In the vocabulary of propositional calculus, only single letters are well-formed formulas. Case 2. C_2 is *wff*. Since $s_1 = sl \quad wff \quad \mathbf{S}$, $s_1^* = sl \quad C_1 or wff \quad \mathbf{S}^*$.

Since C_1 is not *sl*, s^* is false for \mathbf{L} . Substituting a single letter for another symbol in the vocabulary of propositional calculus in a well-formed formula results in a non-well-formed expression. Case 3. C_1 and C_2 are *neg* and *arrow*. Since $s_2 = lpar \quad wff \quad arr \quad wff \quad rpar \quad wff$

\mathbf{S} , $s_2^* = lpar \quad wff \quad aneg or rrr \quad wff \quad rpar \quad wff \quad \mathbf{S}^*$, but s_2^* is false for \mathbf{L} . Since negation and arrow have a different n-arity, they cannot substitute for each other in a well-formed formula. Notice that if the language included, besides negation and implication, symbols for other propositional operators, it would not be necessary to include a new grammatical category for each one. They would be grouped by their n-arity. For example, in the syntax of such an extended language for propositional grammar, ‘ ’ \sim_G ‘ ’ if G is not

redundant. Symbols of the same n-arity can substitute for each other, without affecting their truth or construction. Finally, case 4, C_1 is *lpar* or *rpar*. Since $s_2 = lpar \quad wff \quad arr \quad wff \quad rpar$

$wff \quad \mathbf{S}$, $s_2^* = C_1 or lpar \quad wff \quad aneg or rrr \quad wff \quad rpar \quad wff \quad \mathbf{S}^*$, but s_2^* is false for \mathbf{L} . Also, since $s_2 = lpar \quad wff \quad arr \quad wff \quad rpar \quad wff \quad \mathbf{S}$, $s_2^* = lpar \quad wff \quad aneg or rrr \quad wff \quad C_1 or lpar$

$wff \quad \mathbf{S}^*$, but s_2^* is false for \mathbf{L} . End of claim 3. \sim

3. Trivial Grammars

Definition 1.21 [trivial grammar]: Let $S = \langle S, I, C \rangle$ be a conventional grammar. If $e, f \in E$ ($e \sim_s f$) ($e = f$), then S is *trivial*.

Theorem 1.23: Every language L has a trivial grammar.

Proof: To easily create a trivial grammar for a language, construct for every expression in the language a unique category whose interpretation is its singleton. This specifies that for every expression $e \in E$, $[e] = \{e\}$. For the grammatical rules, constructing a rule for the decomposition of every expression into its basic words is sufficient. For example, for the expression ‘second world war’, construct the rule $\text{SECOND WORLD WAR} \rightarrow \text{SECOND-WORLD-WAR}$, where the only expression in the category SECOND is ‘second’, the only expression in WORLD is ‘world’, the only expression in the category WAR is ‘war’ and the only expression in the category SECOND-WORLD-WAR is ‘second world war’. The only category which may include more than one expression is the category of acceptable string. Hence, the acceptable string ‘The Ocean is Deep’ only needs the inclusion in the grammatical theory of the rule $\text{THE OCEAN IS DEEP} \rightarrow \text{WFF}$. It is straightforward to see that this method has application in any language. \sim

III. Wittgenstein’s Approach

In Wittgenstein’s grammatical method, categories do not depend directly on the rules for building acceptable strings, but proceed from given acceptable strings through allowable substitutions. Two expressions belong to the same grammatical category if they can substitute for each other without affecting the grammar of the expression. Nevertheless, Wittgenstein’s writing is ambiguous as to whether the substitution of expressions belonging to the same category must respect acceptability or all the grammatical categories. The following

pages explore the first interpretation, where the grammar of a word determines the words which can substitute for it preserving acceptability. For the rest of this chapter, two expressions are Wittgenstein-equivalent if the substitution for each other preserves grammatical correctness in any context.²⁰ Substituting some or all of the occurrences of an expression in an acceptable string by another one with the same grammar results in an acceptable expression.

Definition 2.1 [context]: Given a language $\langle \Sigma, E, W \rangle$, let expression e occur in acceptable string w . Omitting an occurrence of e from w and leaving blanks in its place produces an incomplete string called a *context*. A context is not a complete expression, because it includes blank spaces.

Definition 2.2 [Wittgenstein category]: Let C be a context, and let function A assigns to each expression $e \in E$ the string resulting from placing e in the blanks of C . Define the associated *Wittgenstein category* of the context as $B = \{ e \in E \mid A(e) \in W \}$, also expressed as $B = \{ x \in E \mid A(x) \in W \}$.

Definition 2.3 [Wittgenstein-grammatical equivalence]: Let \mathbf{A} be the set of Wittgenstein categories of the language, containing every category associated with any context in the language (resulting from W and E). Given \mathbf{A} , for all $e, f \in E$, $(e \sim_{\mathbf{A}} f)$ iff $\{ C \in \mathbf{A} \mid e \in C \} = \{ C \in \mathbf{A} \mid f \in C \}$. This defines the relation of *Wittgenstein-grammatical equivalence* among expressions. Thus, the above relation's equivalence classes are the Wittgenstein-grammatical categories.

$$[e]_{\mathbf{A}} = \{ A \in \mathbf{A} \mid e \in A \}$$

Proposition 2.4: For all $e, f \in E$, $(e \sim_{\mathbf{A}} f)$ iff $\forall A \in \mathbf{A} [A(e) \in W \iff A(f) \in W]$. \sim

²⁰. BT §9, p.34.

Proposition 2.5: \sim_A is an equivalence relation on E . \sim

IV. A Comparison between the two Approaches

After defining both approaches, it is crucial to determine whether they yield different grammatical categories. This section shows that it is possible to construct a conventional grammar out of Wittgenstein's categories. The rest of this section compares this sort of grammar with other more traditional ones. Most of all, it shows how the distinctions resulting from Wittgenstein's approach to grammar are less fine than the conventional ones.

A. Wittgenstein Grammar

Definition 3.1 [Wittgenstein Grammar and Pure Wittgenstein Grammar]: Let $G = \langle C, I, S \rangle$ be a conventional grammar for a language $L = \langle \Sigma, I, C \rangle$ such that for every pair of expressions $e, f \in E$, $e \sim_G f$ iff $e \sim_A f$, where A is the set of Wittgenstein categories for L . Call G a *Wittgenstein grammar* for L . Let S be a conventional grammar for language L such that $I[C] = A$, where A is the set of Wittgenstein categories for L , then G is a *pure Wittgenstein grammar* for L .

A conventional grammar bears Wittgenstein's name if it respects the notion of Wittgenstein equivalence. It is a *pure Wittgenstein grammar* if the categories interpretations are the language's Wittgenstein categories.

Proposition 3.2: Every pure Wittgenstein grammar is a Wittgenstein grammar. \sim

Proposition 3.3: A conventional grammar G is a Wittgenstein grammar iff $e, f \in E$, $e \sim_G f$ iff $e \sim_W f$, where W is the set of Wittgenstein categories. \sim

Theorem 3.4: Not every language has a pure Wittgenstein grammar.

Proof: Consider the following language L :

$$= \{a, b, c\}$$

$$W = \{ab, ac, ba, bc, ca, cb\}$$

$$E = \quad W$$

The contexts of the language, with its corresponding Wittgenstein categories, are

$A = \{b, c\}$ corresponding to contexts $x(xa)$ and $x(ax)$,

$B = \{a, c\}$ corresponding to contexts $x(xb)$ and $x(bx)$,

$C = \{a, b\}$ corresponding to contexts $x(xc)$ and $x(cx)$,

and W .

Assume towards a contradiction, that L has a pure Wittgenstein grammar $\langle S, I, C \rangle$. $A_1 A_2$

W is a rule $s \quad S$, where A_1, A_2 and $W \quad A$. Unfortunately, every combination of two categories in A yields a string of symbols not in W . A and B produce 'cc', A and C yield 'bb' and B and C make 'aa'. Hence, it is impossible that s be true in L . \sim

The method for producing trivial grammars from Theorem 1.23 generates Wittgenstein grammars for any language. Considering Wittgenstein equivalence classes as grammatical categories guarantees that for every expression $e \in E$, $e \sim_G f$ iff $e \sim_w f$. Constructing a rule for every decomposition of every acceptable string produces a true grammatical theory. For example, if the grammar contains the acceptable string 'rabbits run wild', introduce RABBITS, RUN and WILD as categories and RABBITS RUN WILD \quad WFF as a rule. Assign the interpretation $[\text{rabbit}]_w$ to category RABBITS, the interpretation $[\text{run}]_w$ to RUN, $[\text{wild}]_w$ to WILD, and the set of acceptable strings in the language to WFF.

When using this method to construct a trivial Wittgenstein grammar, including decomposition into basic words, as well as into other complex expressions, is essential. For

example, if the language includes 'run wild' as a complex expression, add to the grammatical theory the rule RABBITS RUN-WILD \sim WFF, where the equivalence class $[\text{run wild}]_w$ interprets the category RUN-WILD. Otherwise, the rules might not use all the Wittgenstein categories.

Clearly, this method applies to any language. However, proving that every language has a Wittgenstein grammar requires the following lemma:

Lemma 3.5: Given a language L , let e and f be a pair of expressions of the language such that f occurs in e . Let g be the string resulting from substituting every occurrence of f in e by h . For any strong grammar S for the language L , if $f \sim_s h$, then $e \sim_s g$.

Proof: Assume, towards a contradiction, that $f \sim_s h$, but not $e \sim_s g$. Then, $e \notin I(C)$ and $g \notin I(C)$ for some conventional grammatical category $C \in C$. Since S is strong, there is a construction P of e as C . Let $P = \langle p_1, p_2, \dots, p_n \rangle$ and let $\langle C_1, C_2, \dots, C_n \rangle$ be a code for it. For every expression p_i occurring in P , construct q_i substituting all occurrences (if any) of f in p_i by h .

Claim: For all $i \leq n$, $q_i \in I(C_i)$.

Proof of claim by induction on i .

Base: $i=1$. Hence, $p_1 \in C_1$. Since no word in the vocabulary occurs in another simple word, either $p_1 = f$ or not. In the first case, if $p_1 = f$, then $p_1 = q_1$. Since $p_1 \in I(C_1)$, then $q_1 \in I(C_1)$. In the second case, if $p_1 \neq f$, then $q_1 = p_1$. Also, $C_1 = C$. Therefore, since $f \sim_s h$, $q_1 \in I(C_1)$. In either case, $q_1 \in I(C_1)$.

Inductive hypothesis: Assume that, for all $j < i$, $q_j \in I(C_j)$, to prove that $q_i \in I(C_i)$.

Either p_i or not. In the first case, $p_i = f \sim_s h = q_i$, so it reduces to the base case. In the second case, some rule $s = D_1 D_2 \dots D_m C_1$ in \mathbf{S} decomposes p_i into some previous expressions in P . In other words, p_i is the result of applying s to a sequence of expressions $\langle e_1, e_2, \dots, e_m \rangle$ such that for all $1 \leq g \leq m$, $e_g = p_k$ and $D_g = C_k$ for some $k < i$. This also means that every expression e_g in $\langle e_1, e_2, \dots, e_m \rangle$ occurs in P somewhere before p_i . Because of this, the inductive hypothesis applies to them. In consequence, for all $1 \leq g \leq m$, $q_k \in I(C_k)$ for some $k < j$. In this case, let $e_g' = q_k$. Since C_k is the category of term D_g in s , rule s applies to $\langle e_1', e_2', \dots, e_m' \rangle$. Since the only difference between e_g and e_g' is the substitution of f for h , q_i is the result of applying s to $\langle e_1', e_2', \dots, e_m' \rangle$. Finally, since C_1 is the resulting category of s , q_i also belongs to the interpretation of C_1 . End of proof of claim.

From the claim, for all $i \leq n$, $q_i \in I(C_1)$. In particular, $p_n \in I(C_n)$. However, $p_n = g$ and $C_n = C$, so $g \in I(C)$, which contradicts the hypothesis. \sim

Corollary 3.6: Let $e = e_1 \{e_2 \{ \dots \{e_n$, and $e' = e_1' \{e_2' \{ \dots \{e_n'$. Given any strong grammar \mathbf{S} , if, for all $1 \leq i \leq n$ $e_i \sim_s e_i'$ then $e \sim_s e'$. \sim

Corollary 3.7: Let e be an acceptable string of a language \mathbf{L} such that another expression f occurs in e . Let g be the string resulting from substituting every occurrence of f in e by h . For any conventional grammar \mathbf{S} for the language \mathbf{L} , if $f \sim_s h$, then g is acceptable too. \sim

Corollary 3.8: Given a language $\langle \Sigma, E, W \rangle$, let $e = e_1 \{e_2 \{ \dots \{e_n$ W , and $e' = e_1' \{e_2' \{ \dots \{e_n'$ E . Given any conventional grammar \mathbf{S} , if, for all $1 \leq i \leq n$ $e_i \sim_s e_i'$ then $e' \sim W$.

Theorem 3.9: Every language L has a Wittgenstein grammar.

Proof: Let $L = \langle \Sigma, E, W \rangle$ be a language. Let C be the set of Wittgenstein-equivalence classes. $C = \{ [e]_w \mid e \in E \}$. Let I be the identity function. Let D be the set of tuples of (more than one) expressions whose concatenation is also an expression of the language. In other words, for all $e_1, e_2, e_3, \dots, e_n \in E$; $\langle e_1, e_2, e_3, \dots, e_n \rangle \in D$ iff $e_1 \{e_2 \{e_3 \{ \dots \{e_n \in E$, where $n > 1$. Let $S = \{ [e_1]_w [e_2]_w [e_3]_w \dots [e_n]_w \mid [e_1 \{e_2 \{e_3 \{ \dots \{e_n]_w \mid \langle e_1, e_2, e_3, \dots, e_n \rangle \in D \}$.

Claim 1: $e \in E, e \sim_s f$ iff $e \sim_w f$.

Proof: Assume $e \sim_s f$ to prove $e \sim_w f$. Since $e \sim_s e, f$ must also belong to $[e]_w$. This means that $e \sim_w f$. For the converse, assume e is not grammatically equivalent in S to f to show that they are not Wittgenstein equivalent either. Without loosing generality, $f \notin [e]_w$. But, in any case, $f \in [f]_w$. Hence, $[e]_w \neq [f]_w$. In other words, it is false that $e \sim_w f$.

Claim 2: $\langle S, I, C \rangle$ is a grammar for L .

Proof: Assume not. $\langle S, I, C \rangle$ may not be a grammar only if I does not model S . Then, some rule $s = [e_1]_w [e_2]_w [e_3]_w \dots [e_n]_w \mid [e_1 \{e_2 \{e_3 \{ \dots \{e_n]_w$ S is not true for the language L . An n -tuple of language expressions $\langle e_1', e_2', e_3', \dots, e_n' \rangle$ does not satisfy s . Even though, for all $i \leq n$, $e_i' \in [e_i]_w$, $e_1' \{e_2' \{e_3' \{ \dots \{e_n' \in [e_1 \{e_2 \{e_3 \{ \dots \{e_n]_w$.

Nevertheless, from Corollary 3.8, since $e_1, \dots, e_n \in W$ and for all $i \leq n$, $e_i \sim_w e_i$, it must also be the case that $e_1, \dots, e_n \in W$. This contradiction proves that $\langle S, I, C \rangle$ is a grammar for L . Since $e \in E$, $e \sim_s f$ iff $e \sim_w f$, $\langle S, I, C \rangle$ is also a Wittgenstein grammar. \sim

B. Formal Requirements for a Wittgenstein Grammar

The previous section demonstrated the existence of Wittgenstein grammars. This section displays the formal constraints on Wittgenstein grammars. It compares traditional grammatical categories with Wittgenstein's. It reveals the sort of grammars for which Wittgenstein's approach produces finer (or at least as fine) categorical distinctions. The traditional relation of grammatical equivalence does not always match that of Wittgenstein equivalence. It is false that, for every language and every grammar $e \sim_s f$ iff $e \sim_w f$. This section shows the structural features of the grammar falsifying the double implication. It describes the sort of grammars where Wittgenstein equivalence implies grammatical equivalence, and vice versa.

Definition 3.10 [normal grammar]: A conventional grammar $\langle S, I, C \rangle$ is *normal* for a language $\langle \Sigma, E, W \rangle$ iff, for every expression $e \in E$ and every acceptable string $w \in W$, if e occurs in w , e occurs in a tree for w in $\langle S, I, C \rangle$ as many times as e occurs in w .

Lemma 3.11: Let A be a context in a normal language $L = \langle \Sigma, E, W \rangle$. Given a strong conventional grammar $\langle S, I, C \rangle$, for every pair of acceptable strings $A(e)$ and $A(f)$, there is a pair of trees T_e and T_f isomorphic down to a node t_e such that for all $t \in T_e$ there is a $t' \in T_f$ such that (i) $\text{label}(t)$ results from $\text{label}(t')$ when substituting f for e once at most such

that (ii) the category of $\text{label}(t)$ in the proof corresponding to \mathbf{T}_e is the same as the category of $\text{label}(t')$ in the proof corresponding to \mathbf{T}_f .

Proof: Since \mathbf{L} is a normal language, there is a tree \mathbf{T}_e of $A(e)$ such that e occurs as many times in \mathbf{T}_e as e occurs in $A(e)$. Let P_e be the proof corresponding to tree \mathbf{T}_e . First, define the function $\text{Rule} : \mathbf{T}_e \rightarrow \mathbf{S}$ mapping every node t in \mathbf{T}_e to the rule in the grammatical theory \mathbf{S} such that $\text{label}(t)$ occurs in P_e in virtue of s and the function $\text{Cat} : \mathbf{T}_e \rightarrow \mathbf{S}$ maps every node t in \mathbf{T}_e to the corresponding category in the code of P_e .

Let t_e be a node in \mathbf{T}_e such that $\text{label}(t_e) = e$. Let P_f be the construction of f as $\text{Cat}(t_e)$, and \mathbf{T}'_f be its tree, with $<'_f$ as its ordering relation and label' as its labeling function. Since $\langle \mathbf{S}, \mathbf{I}, \mathbf{C} \rangle$ is strong, this construction exists. Let $\mathbf{T}_f = \mathbf{T}'_f \cup \mathbf{T}_e$. Define function $f : \mathbf{T}_f \rightarrow \mathbf{E}$ the following way: (i) $f(t) = \text{label}'(t)$ if $t \in \mathbf{T}'_f$, (ii) $f(t) = \text{label}(t)$ if t does not belong to \mathbf{T}'_f and e does not occur in $\text{label}(t)$, (iii) otherwise, recursively define $f(t_e) = f$ and for all $t > t_e$ let t be such node in \mathbf{T}_e that e occurs in $\text{label}(t)$ and let u be the least unique node in \mathbf{T}_e such that $t < u$. $\text{Label}(u)$ is the result of applying $\text{Rule}(u)$ to an n -tuple of expressions $\langle e_1, e_2, e_3, \dots, e_n \rangle$, such that, for every expression $e' \in \mathbf{E}$, $e' = \text{label}(t')$ for some t' in \mathbf{T}_e and u is the least node such that $t' < u$ iff $e' = e_i$ for some $i \leq n$. Since u is the least node such that $t < u$, $\text{label}(t)$ occurs in the n -tuple $\langle e_1, e_2, \dots, \text{label}(t), \dots, e_n \rangle$. However, $\text{label}(t)$ may well occur more than once in the n -tuple. Since $\text{Rule}(u)$ is of the form $C_1 C_2 \dots C_n \rightarrow C$, there is an $i \leq n$ such that $C_i = \text{Cat}(t)$ and $e_i = \text{label}(t)$. Actually, more than one may satisfy these two conditions. Which one the proof uses makes no difference.

Now, since $f \in I(\text{Cat}(t)) = I(C_i)$, $\text{Rule}(u)$ applies to the n -tuple $\langle e_1, e_2, \dots, e_{i-1}, f(\text{label}(t)), e_{i+1}, \dots, e_n \rangle$ resulting from substituting $\text{label}(t)$ by $f(\text{label}(t))$ in the i th place. Hence $e_1 \{ e_2 \{ \dots \{ e_{i-1} \{ f(\text{label}(t)) \{ e_{i+1}, \dots \{ e_n \} \} \} \} \} \in I(\text{Cat}(u))$. Then, let $f(e_1 \{ e_2 \{ \dots \{ e_{i-1} \{ \text{label}(t) \{ e_{i+1}, \dots \{ e_n \} \} \} \} \} = e_1 \{ e_2 \{ \dots \{ e_{i-1} \{ f(\text{label}(t)) \{ e_{i+1}, \dots \{ e_n \} \} \} \} \}$. Since $\text{label}(t)$ and $f(\text{label}(t))$ differ only in one substitution of e by f , so do $(e_1 \{ e_2 \{ \dots \{ e_{i-1} \{ \text{label}(t) \{ e_{i+1}, \dots \{ e_n \} \} \} \} \}$ and $f(e_1 \{ e_2 \{ \dots \{ e_{i-1} \{ \text{label}(t) \{ e_{i+1}, \dots \{ e_n \} \} \} \}$.

Finally, consider the tree $\langle \mathbf{T}_f, \prec_f \rangle$ where $\prec_f = \prec'_f \prec \sim (t \prec_e t_e) \quad \{(\text{top}(\mathbf{T}'_f), t_e)\}$ is the ordering relation, and $f(\text{label}(t))$ labels every node t in \mathbf{T}_e . Since $f(\text{label}(t)) \in I(\text{Cat}(t))$ for all t in \mathbf{T}_e , the result of re-labeling \mathbf{T}_e is also a tree. It is a tree proving that $f(A(e))$ is acceptable. Since $f(A(e))$ differs from $A(e)$ in one substitution of f for e , $A(e) = A'(e)$ and $f(A(e)) = A'(f)$ for some context A' . The possibilities of repeating this process is the same as the number of times e occurs in $A(e)$. Each one provides a different context A' . Also, since erasing one time e occurs in $A(e)$ produces every context, this is the same number of contexts A' for which $A'(e) = A(e)$. In particular, A' must equal A . \sim

Theorem 3.12: If a strong conventional grammar \mathbf{G} is normal for a language $\mathbf{L} = \langle \Sigma, E, W \rangle$, then for all $e, f \in E$, $(e \sim_{\mathbf{G}} f) \iff (e \sim_w f)$.

Proof: Let $\mathbf{G} = \langle \mathbf{S}, \mathbf{I}, \mathbf{C} \rangle$ be a strong conventional grammar normal for language $\langle \Sigma, E, W \rangle$. Assume, towards a contradiction, that $e \sim_{\mathbf{G}} f$ but not $e \sim_w f$. This implies that e and f belong to all the same categories in \mathbf{C} , but not in \mathbf{A} . Without loss of generality, $e \in A$, and $f \notin A$ for some Wittgenstein category $A \in \mathbf{A}$. In consequence, $A(e) \in W$ but $A(f) \notin W$. Now, since $A(e) \in W$ and \mathbf{G} is normal, e occurs in the tree T of a proof P of $A(e)$ as many times

as e occurs in $A(e)$ itself. Furthermore, by lemma 3.11, it is possible to substitute e for f only the e in $A(e)$ yielding context A . This strategy creates a tree and proof of $A(f)$. Nevertheless, this would mean that $A(f)$ is an acceptable string, contradicting the hypothesis that $A(f) \notin W$. \sim

However, an ambiguity remains in Wittgenstein's thesis saying two expressions belong to the same grammatical category, if they can stand in place of each other in some context without affecting the grammar of the original expression. The preceding section dealt only with one possible interpretation of this thesis. In such an interpretation, contexts result from well-formed sentences. Accordingly, the substitution of grammatically equivalent expressions preserves the acceptability of statements. Still, another interpretation is possible. A stronger relationship of grammatical equivalence results from allowing the production of contexts from any grammatical phrase, instead of only full sentences. The resulting relation of equivalence is stronger, because it respects all the grammatical categories. In contrast, this chapter's approach respected only the category of well-formed sentence. Under this alternative interpretation, two expressions would be grammatically equivalent if they could substitute for each other within any expression without affecting the expression's grammatical categories. Substituting some or all the times an expression occurs in any well-formed expression – not necessarily a complete sentence – with a synonymous expression must result in an expression belonging to exactly the same categories as the original. The results from this sections are bound by considering only the first interpretation. Generalizing these results would require performing a more thorough investigation into the relation between these grammars and natural language.

V. Conclusion

Two final conclusions summarize the results of this chapter. First, it is not possible to construct conventional grammars for every language out of the categories resulting from Wittgenstein's analysis. Nevertheless, it is always possible to construct a 'Wittgenstein grammar' where grammatical equivalence corresponds to Wittgenstein equivalence. This justifies calling 'grammar' whatever results from the kind of analysis of substitutions Wittgenstein proposes. On the other hand, this sort of grammar does more than satisfy the minimal expectations for a grammar. Wittgenstein grammars are not always strong. They do not provide enough information to construct all acceptable expressions out of the basic words in the language. In this sense, they are weaker than other conventional grammars. Nevertheless, they are not the weakest. In general, Wittgenstein's grammatical distinctions are neither the most specific nor the most general.

Wittgenstein's philosophy provides no straightforward interpretation of these formal results. It is tempting to dismiss Wittgenstein's notion of grammar as too weak to play the central role he expects it to.

This chapter formalized some of Wittgenstein's intuitions about grammar and drew several philosophical conclusions from them. Now, it is time to place them in the bigger picture of Wittgenstein's philosophy of mathematics. The following chapter builds on the results of this analysis. It applies the previous formal reconstruction to a portion of language containing numerical expressions. It proves that the grammatical theory resulting from Wittgenstein's approach contains expressions whose natural interpretation is mathematical. It shows that arithmetical rules regulate the use of numerical expressions in natural language.

Chapter 6

Mathematics as Grammar

A formal treatment

I. Introduction

The question, ‘Is mathematics part of the formal grammar of language?’ has been part of the philosophical debate on mathematics for more than a century. It lies at the center of the debate between Carnap and Bar Hillel, on the one side, and Gödel, Tarski and Quine on the other. The previous two chapters provided the background for an answer to those questions. The third chapter explained the relationship between mathematics and natural language, while the fourth one provided a formal model of grammatical analysis. An accurate picture of the relationship between mathematical calculus and the grammar of natural language develops only through the combination of the results from those two chapters.

A. A Formal Model of Grammar

Taking seriously Wittgenstein’s claim that mathematical numerical expressions are grammatical demands a clear understanding of Wittgenstein’s definition of ‘grammar’. At first, Wittgenstein’s notion of grammar does not seem to be that of common use. The absence of an explicit definition of grammar in his published writings makes it difficult to tackle Wittgenstein’s stance on such questions as ‘What is the relationship between mathematical calculi and the grammar of natural language?’ In the *Big Typescript*, Wittgenstein briefly offers an explanation of ‘grammar’ which lacks specificity. Remaining faithful to Wittgenstein requires sticking to the textual evidence in evaluating his grammatical claims. However, this commitment does not counter using a formal understanding of grammar

when working within the limits of what Wittgenstein explicitly wrote about grammar in this period. The previous chapter provided a precise formalized theory of grammatical analysis fundamental for understanding its interactions with mathematics.

B. *Anwendung* and the Grammar of Natural Language

The third chapter, on the notion of *Anwendung*, explained the role calculations play in the solution of practical problems. For Wittgenstein, mathematics consists entirely of calculations [*Rechnungen*]. Calculations cannot solve anything but mathematical problems. Mathematical calculations are used all the time to solve practical problems. However, even though they are essential to the solution of some practical problems, mathematical calculations do not justify predictions about affairs outside the calculus to which they belong. Therefore, investigating the relationship between mathematical calculation and factual prediction offers more insight than asking about the relationship between mathematical and empirical propositions.

Calculations also provide the grammar of predictive statements. For example, dividing twelve by three in the arithmetic of natural numbers furnishes the grammar of the statement ‘if I have eleven apples, I can share them among three people in such a way that each is given three apples’. Calculations give solutions to practical problems just as they do mathematical ones: they provide a grammatical rule whose applications are propositions. These propositions may occur either inside or outside the calculus in internal and external applications of the rules that constitute a mathematical calculus.

Another important conclusion from the third chapter was that mathematical expressions, like numerals, have the same meaning in natural language as in pure mathematics. Their grammar is the same. In particular, arithmetic provides the grammar of cardinal nume-

als in natural language. It is part of the grammar of natural language. This fifth chapter tests this thesis within the formal framework developed in the previous chapter.

C. Mathematics as Grammar

This fifth chapter proves that a grammatical analysis applied to natural language produces propositions with a naturally mathematical interpretation. The notion of *Anwendung*, as developed in chapter 4, grounds Wittgenstein's thesis that the grammatical analysis of a portion of language which is the application of a mathematical calculus results in a grammar identical to the calculus. For Wittgenstein, a mathematical calculus has two different sorts of *Anwendung*: an internal one in its very own calculations and an external one in natural language (or another calculus). Reflecting this, this chapter grammatically analyzes both the internal and external *Anwendung* of elementary arithmetic.

For the first task, consider the arithmetical calculus as a language where the correct calculations are the acceptable expressions. Taking natural language as given – where the acceptable expressions are none other than the grammatically correct statements – is sufficient for the second task. In both cases, it not only produces adequate arithmetical propositions, but also demonstrates the adequacy of Wittgenstein's philosophical interpretation of such propositions. In other words, this shows, first, that mathematical equations result from the grammatical analysis of the traces the arithmetical operations left behind and, second, that such equations express a connection between the numbers and operations as grammatical categories.

The first stage of such demonstration is a formal grammatical study of the notion of natural number. The first step defines numbers as grammatical categories. Adhering to Wittgenstein's philosophical theory of numbers, they are defined as the results of particular additions. Every addition results in a single number. The second step shows that numbers so

defined are in fact classes of interchangeable expressions. Contexts of intersubstitutability in the language of arithmetical calculation define grammatical categories. Finally, the concluding step shows that the traditional recursive definition of natural number is equivalent both to the definition of numbers as the result of additions and to the conception of numbers as grammatical equivalence classes of numerals.

This requires showing that, for every natural number in the traditional sense, the Wittgensteinean grammar of arithmetical calculus possesses a category containing only the appropriate numerals. It is also necessary to show that, if the induction principle holds for them, they form the smallest set of categories, closed under successor, including zero. This requires defining the system of cardinal numbers through grammatical analysis. This analysis shows that a unique grammatical category for ‘one’ exists at the base of the cardinal numerical system. It also defines the function of successor in grammatical terms and translates the induction principle to the grammatical vocabulary developed in the previous chapter.

II. A Grammatical Analysis of the Internal *Anwendung* of Arithmetic.

Wherever in a calculus one number can be replaced by another . . . They are the same.

WL §8 p. 218

A. Introduction

This section shows how internal *Anwendung* works within a calculus. The calculus in question is the subsystem of elementary arithmetic containing elementary additions of natural numbers in base ten notation. It applies last chapter’s formal model of grammatical analysis to this subsystem of arithmetical calculation .

1. Calculi as Languages

The first phase consists of fitting the arithmetical subsystem in the previous chapter's definition of language. For this purpose, the digits one to ten and the addition and equal signs define the alphabet of the language. Particular additions, that is, numerals linked by the addition sign, play the role of calculation signs in this calculus, and equations constitute the acceptable strings. Additions, equations and numerals make up the vocabulary of the language.

2. A Grammatical System of Natural Numbers

Three different accounts of 'number' appear in Wittgenstein's writings during the early thirties. The following section provides, within the framework of grammar, an explicit formulation of such accounts and then demonstrates their equivalence. Wittgenstein explicitly offers the first two accounts, while the third reformulates the traditional definition of a system of natural numbers.

1. A natural number is the result of an arithmetic operation, in this case: addition.

2. A natural number is the grammatical-equivalence class of a numeral, the class of all words interchangeable with a numeral throughout the language.

3. The set of natural numbers is the smallest set to include zero and be closed under successor.

(1) is **Definition 4.2.3. Corollary 4.2.3.1.** demonstrates the implication from definitions (1) to (2.) Finally, **Theorem 4.4.4.** shows the equivalence between definitions (2) and (3).

3. The Grammar of Addition

This section shows how a grammatical analysis of Wittgenstein's sort produces correct mathematical equations. After all, language analysis starts with a definition which naturally

includes the class of correct equations. It would not be surprising if they were also the result of such analysis. The grammatical analysis would amount to less than a sleight of hand. However, this is not so. It is true that the beginning material includes traces of correct calculations as ‘given’. Nevertheless, the mathematical equations are results in the meta-language. They are explicitly grammatical rules about numbers, where numbers are grammatical categories. For example, that $3 + 4 = 7$ is a theorem of the grammar means that adding expressions belonging to the grammatical categories ‘number three’ and ‘number four’ results in an expression belonging to the grammatical category ‘number seven’. Equations play a double role in the calculus. They are both acceptable configurations in the language, and grammatical rules of the calculus. The introduction of a meta-language serves the purpose of making this double role explicit.

This section is important for two reasons. On the one hand, it supports the original hypothesis that Wittgenstein’s grammatical method of analysis gives results with a natural mathematical interpretation. Also, it justifies this research’s interpretation of Wittgenstein’s philosophy of mathematics. Just as the first part demonstrates that numbers are grammatical categories of inter-substitutable numerals, the second part shows that arithmetical equations connect calculations – in this case, additions – with their results.

B. A Formal Grammatical Analysis of a Subsystem of the Arithmetic of Natural Numbers with Addition as its Single Operation

This section follows the following notational conventions: Capital Latin letters stand for grammatical categories. Bold Greek upper case letters stand for other sets of expressions. Lower case Latin letters stand for particular expressions of the language, except for italic ones. Italic lower case Latin letters stand for variables in the definition of categories through

lambda abstraction. A cursive upper case letter L stands for the language structure. \mathbb{N} stands for the set of numbers. All variables range over expression *types* and sets thereof and do not refer to expression tokens at all.

1. The Calculus as Language

Definition 4.1.1 [digits]: Let $\Delta = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of *digits*.

Definition 4.1.2 [numerals]: Let \mathbf{N} be the set of all *numerals* in base ten notation, finite sequences of digits not starting with a numeral zero.

Definition 4.1.3 [additions]: Let \mathbf{A} be the set of *additions* as recursively defined on the base of the numerals so that $\mathbf{A} = \{ a \text{ '+' } b \mid a, b \in \mathbf{N} \}$. Notice that avoiding parentheses builds associativity into the language.

Definition 4.1.4 [numerical expression]: Any expression different from '+' and '=' is called *numerical*.

Proposition 4.1.4.1: Every addition is a numerical expression. \sim

Proposition 4.1.4.2: Every numeral is a numerical expression. \sim

Proposition 4.1.4.3: Every numerical expression is either a numeral or an addition. \sim

Definition 4.1.4 [addition operator on numerical expressions]: For every two numerical expressions a and b , $a + b = a \text{ '+' } b$.

Proposition 4.1.4.1: For every two numerical expressions a and b , $a + b \in \mathbf{A}$. \sim

Definition 4.1.5 [language]: Define language L by the structure $\langle \Sigma, \mathbf{E}, \mathbf{W} \rangle$, where $\Sigma = \Delta \cup \{+, =\}$ is its alphabet, \mathbf{W} is the class of traces of correct calculations, that is, equations such as '3 + 4 = 7' or '2 + 2 = 4',¹ and $\mathbf{E} = \mathbf{W} \cup \mathbf{N} \cup \mathbf{A}$ is the set of *basic*

¹. The acceptable strings are not true statements of arithmetic, but the final traces of correct additions. This means that expressions like '3 + 4 = 7' are included, but '7 = 7' and '3 + 4 = 4 + 3' are not. Cf. Chapter 2, section III B.

words or expressions of the language. This definition satisfies the three conditions of Definition 1.1 in Chapter 4: (1) Σ is a finite, non-empty set, (2) W and E are subsets of the set of finite strings of such words such that $(\sum W) \subseteq E$ and (3) every member of E is a substring of some element of W . In other words, every number occurs in a true addition equation.

2. Grammatical Number Theory

Definition 4.2.1 [result operator]: Given an addition $a \in A$, define the category $R(a) = \{x \mid (a \text{ '=' } x)\}$ as the *result* of addition a .

Definition 4.2.2 [number]: A *number* is any category of the form $\{x \mid (a \text{ '=' } x)\}$ where $a \in A$. In other words, a category is a number if it is the result of some addition.

Proposition 4.2.2.1: For any number category C and numeral n , if $n \in C$, then $C = [n]$.

Proof: Suppose $n, m \in C$ to show that $n \sim m$. n and m may occur either at the right or the left of the '=' sign. For every addition a , $a \text{ '=' } n \in W$ iff $a \text{ '=' } m \in W$. This guarantees that m may replace any occurrence of n at the right of '='. Furthermore, let a be one of such additions. In every addition different that a itself, a is substitutable for n or m . Also, in any addition, n and m are substitutable for a . By transitivity, n is substitutable for m in any addition. In consequence, they are interchangeable at either side of '='.

3. The Grammar of Addition

Definition 4.3.1 [addition category of a numeral]: For every numeral n , let the *addition category* of n , written $A(n)$, be the category $\{x \mid (x \text{ '+' } n)\}$. This is the category of all additions – numerals linked by the addition sign '+' – whose correct result is n .

Proposition 4.3.1.1: For every a in $A(n)$, $n \in R(a)$. \sim

Lemma 4.3.2: For every pair of numerals n, m in \mathbf{N} , if there is a number N such that $n, m \in N, A(n) = A(m)$.

Proof: Let N be a number, such that $n, m \in N$. Also, let $a \in A(n)$. From the previous corollary, $n \in R(a)$. Hence, $R(a)$ is a number such that n belongs to it. From theorem 4.2.3.1, n can only belong to one number, so $N = R(a)$. Hence, $m \in R(a)$. Since $a \in A(n)$, this means that $(a \text{ '=' } m) \in W$, which implies that $a \in A(m)$. \sim

Definition 4.3.3 [addition category of a number]: For any number N , define the *addition category* $A(N)$ as $A(n)$ where $n \in N$. Hence, $A(N) = A(n)$ for all n in N .

Theorem 4.3.4: For any addition a in \mathbf{A} , $A(R(a)) = [a]$.

Proof: Obviously, $a \in (R(a))$. Suppose $b \in (R(a))$ to show that $b \sim a$. Let C be any category C such that $a \in C$, to show that $b \in C$. Since a and b are additions – numerals linked through the plus sign, they occur only to the left of the '=' sign. Without losing generality, C is of the form $x(c \text{ '+' } x \text{ '+' } d \text{ '=' } m)$ where $c, d \in A$ and $m \in N$. This means that $(c \text{ '+' } a \text{ '+' } d \text{ '=' } m) \in W$. Since $b \in A(R(a))$, $R(b) = R(a)$. In consequence, $(c \text{ '+' } b \text{ '+' } d \text{ '=' } m) \in W$. So $b \in x(c \text{ '+' } x \text{ '+' } d \text{ '=' } m) = C$. \sim

Definition 4.3.5 [numerical category]: A category A is *numerical* if for all a and b in A , $R(a) = R(b)$.

Definition 4.3.5.1 [result of a numerical category]: Given a numerical category A , define the *result* of A as $R(A) = R(a \text{ '+' } 0')$ for any a in A .

Definition 4.3.6 [addition of numerals]: Given two numerical expressions a and b , define the *addition* of a and b , written as $A(a + b)$, as the category $x(x \text{ '=' } c)$, where c is an expression in $R(a \text{ '+' } b)$.

Proposition 4.3.6.1: $(a \text{ '+' } b) \in A(a + b)$. \sim

Lemma 4.3.7: Let A and B be two numbers such that $a \in A$ and $b \in B$. For every pair of expressions $a' \in A$ and $b' \in B$, $a' \text{ '+' } b'$ belongs to the addition of a and b .

Proof. Since, as noted above, the only contexts in which numerals of the same number differ is inside numerals, a' may substituted for a and b' for b in their addition without affecting the result. \sim

Lemma 4.3.8. Let A and B be two additions such that $a \in A$ and $b \in B$. For every pair of expressions $a' \in A$ and $b' \in B$, $a' \text{ '+' } b'$ belongs to $A(a + b)$.

Proof: $A(R(a) + R(b)) = A(a + b)$. $A(R(A) + R(B)) = A(a + b)$.

$A(R(a') + R(b')) = A(a + b)$. $A(a' + b') = A(a + b)$. \sim

Theorem 4.3.9: Let A and B be two numerical categories such that $a \in A$ and $b \in B$. For every pair of expressions $a' \in A$ and $b' \in B$, $a' \text{ '+' } b'$ belongs to $A(a + b)$.

Proof: Grammatical categories include only numerals and additions. \sim

Definition 4.3.10. [addition of categories]: Define the addition of numerical categories A and B , written $A + B$, as $A(a + b)$ where $a \in A$ and $b \in B$.

Lemma 4.3.11: $R(R(a) + R(b)) = R(a + b)$.

Proof: The addition of two numerals is not a numeral, but an addition. Hence, the addition of two numbers is not a number either. However, the result of a numerical category is always a number. \sim

Theorem 4.3.12: For every equation of the form $a + b = c$ in W , c belongs to the number which is the result of the addition $[a] + [b]$. In other words, $R([a] + [b]) = [c]$.

Proof: $R(a + b) = R(c)$. Hence, $R([a] + [b]) = [c]$. \sim

Note 4.3.13: Notice that, however, for every equation of the form $a + b = c$ in W , $[a] + [b] = [c]$ is not a theorem of this theory. In other words, the addition of two numbers is not another number. Arithmetical equations are not numerical identity statements. That is precisely the anticipated result from Wittgenstein's analysis of equations, where the '=' sign does not symbolize identity but a connection between calculation and result.

4. A Grammatical System of Natural Numbers

Definition 4.4.1 [zero]: Let 0 be the [Wittgensteinean] category $x ('0 + 0 = ' x)$.

Proposition 4.4.1.1: '0' 0. ~

Definition 4.4.2 [one]: Let 1 be the [Wittgensteinean] category $x ('0 + 1 = ' x)$.

Proposition 4.4.2.1: '1' 1. ~

Definition 4.4.3 [successor]: Define the *successor* function S defined over the numbers such that $S(N)$ is the number $x (n '+' '1' '=' x)$ where n is any expression in N .

Proposition 4.4.3.1: $S(0) = 1$.

Proof: '0 + 1 = 1' W . ~

Theorem 4.4.4: Let $\tilde{}$ be the set of natural numbers. $\tilde{}$ is the smallest set closed under S such that $0 \in \tilde{}$.

Proof: This theorem requires proving: (i) that 0 is an element of $\tilde{}$, (ii) that $\tilde{}$ is closed under successor, and (iii) $\tilde{}$ is the smallest set with these two properties.

i) Prove that $0 \in \tilde{}$, that is, 0 is a number.

Proof of (i): ‘0 + 0 = 0’ $\in \mathbf{W}$ and ‘0 + 0’ $\in \mathbf{A}$. By the definition of *number*, ‘0’ $\in \mathcal{X}$ (‘0 + 0 =’ $\in \mathcal{X}$). For every n in \mathbf{N} , $n = 0$ iff $n \in \mathcal{X}$ (‘0 + 0 =’ $\in \mathcal{X}$), so \mathcal{X} (‘0 + 0 =’ $\in \mathcal{X}$) = 0 $\in \mathcal{X}$.

ii) Prove that if N belongs to \mathcal{X} , so does $S(N)$.

Proof of (ii): Let N be a number, show that $S(N)$ is also a number. Let n be any expression of number N . n ‘+ 1’ $\in \mathbf{A}$. Since $S(N)$ is the category \mathcal{X} (n ‘+’ ‘1’ ‘=’ $\in \mathcal{X}$), it is of the form \mathcal{X} (a ‘=’ $\in \mathcal{X}$) where $a \in \mathbf{A}$. In consequence, $S(N)$ is a number.

iii) Let \mathbf{N}^* be a set of categories such that $0 \in \mathbf{N}^*$ and \mathbf{N}^* is closed under successor. Prove that $\mathcal{X} \subseteq \mathbf{N}^*$. In other words, prove that zero reaches every number N in \mathcal{X} by repeated applications of the successor function.

Proof of (iii): Since every number has the form \mathcal{X} (a ‘=’ $\in \mathcal{X}$) where $a \in \mathbf{A}$, (iii) amounts to the proposition that, for all a in \mathbf{A} , $R(a) \in \mathcal{X}$. Considering only additions consisting of two numerals united by the ‘+’ sign eases the presentation of this proof. However, the proof applies to additions of any finite length, as well. This proof is an induction on \mathbf{A} . Ordering \mathbf{A} is necessary. Consider the following: ‘0 + 0’, ‘0 + 1’, ‘1 + 0’, . . . The category \mathcal{X} (‘0 + 0 =’ $\in \mathcal{X}$), belonging to \mathcal{X} bases this induction. Since, \mathcal{X} (‘0 + 0 =’ $\in \mathcal{X}$) = 0 $\in \mathcal{X}^*$, the base holds. Now, suppose as inductive hypothesis, that for every $a^* < a$, $R(a^*) \in \mathcal{X}^*$, to show that $R(a) \in \mathcal{X}^*$. Since $a = b + c$ for some b, c in either \mathbf{A} or \mathbf{N} , $R(a) = R(b + c)$. If b or c is a numeral, then its grammatical-equivalence class belongs to \mathcal{X} . This is obvious, because the successor function constructs all numerals from zero. If it is an addition then it is before a in the ordering of \mathbf{A} , so the inductive hypothesis applies to it. In either case,

$R(b) \sim^*$ and $R(c) \sim^*$. From Lemma 4.3.11, $R(R(b) + R(c)) = R(b + c)$. Since $R(b) \sim^*$, $b = d + 1$ for some addition d . Hence, $R(a) = R(d + c + 1) = R(R(d + c) + 1)$. But $(d + c) < a$, so $R(d + c) \sim^*$. In consequence, $R(R(d + c) + 1) \sim^*$. Finally, since $R(a) = R(R(d + c) + 1)$, $R(a) \sim^*$.

C. Mathematical Induction

Besides showing the desired hypothesis, the previous formal analysis of arithmetical addition for natural numbers produced another important conclusion. Theorem 5.4.4. shows that Wittgenstein's notion of numbers as grammatical categories form a system of natural numbers. This stands against those who argue that, during this period, Wittgenstein rejected the induction principle for arithmetic. However, this appraisal needs qualification. Wittgenstein rejected the existence of universal mathematical propositions. For him, mathematical propositions of the form $\forall x(x \in \dots)$, where \dots is a mathematical concept, are not about *all* the members of such mathematical category.² Mathematical inductions are not inductions in the sense this word has in natural science. They are calculations. In consequence, mathematical inductions do not prove universal properties of all members of mathematical concepts. Mathematical propositions, whose proof is an induction are not more general than those any other calculation proves. They cannot prove anything general about other calculus.

². Chapter 2, on mathematical concepts, presented Wittgenstein's argument for this heterodox view.

III. A Grammatical Analysis of the External Application of Arithmetic

I want to say numbers can only be defined from propositional *forms*, independently of the question which propositions are true or false.

PR §102

The analysis of Wittgenstein's notion of *Anwendung* in Chapter 3 concluded that mathematics is part of the grammar of natural language. When Wittgenstein says that mathematical propositions are grammatical rules, the expression 'grammatical rules' is not a metaphor. Mathematical rules are equal to obviously grammatical ones like 'adverbs qualify verbs and adjectives, but not nouns'.

Not surprisingly, Wittgenstein's claim has endured scorn and ridicule both from philosophers and linguists. Reactions include responses such as 'mathematicians are not grammarians' and 'there is no way that one may ever learn mathematics from reading the dictionary'. Many people are under the wrong impression that natural language grammar cannot tell different numbers apart, because all numerical expressions share the same grammar. They think that the substitution of numerical expressions *inter alia* does not jeopardize their grammatical correctness. These people think that, in general, saying 'I bought ten chars at the market yesterday' is as correct as saying 'I bought three chairs at the market yesterday', even though ten and three are different numbers. However, as this section will show, they are mistaken. Natural language provides for the grammatical distinction between numbers.

One of the most common philosophical arguments against mathematics being grammar is that mathematical entities challenge the accepted concept of grammatical categories. However, this final section proves the contrary. Numbers, or mathematical entities in general, are grammatical categories in precisely the same sense as adjective, noun, etc. Furthermore, reading the dictionary actually teaches mathematics.

A. A Numerical System.

Was die Zahlen sind? – Die Bedeutung der Zahlzeichen; und die Untersuchung dieser Bedeutung ist die Untersuchung der Grammatik der Zahlzeichen.

Wir suchen nicht nach einer Definition des Zahl-Begriff. Sondern versuchen eine Darlegung der Grammatik des Wortes “Zahl” und der Zahlwörter. [PG PT. II. Section IV §18 p. 630]

What are numbers? – What numerals signify; an investigation of what they signify is an investigation of the grammar of numerals.

What we are looking for is not a definition of the concept of number, but an exposition of the grammar of the word “number” and of the numerals. [PG PT. II - IV §18 p. 321]

The first step translates the conventional definition of a numerical system into the grammatical vocabulary and shows how the cardinal numbers result from the application of Wittgenstein’s grammatical analysis from the early thirties, as formalized in the previous chapter.

1. ‘One’

According to ordinary English grammar, the word ‘one’ belongs to almost every major grammatical category: noun, pronoun, adjective and even verb. Mostly, it appears in the composition of nominal expressions (complex names) such as ‘one friend of mine’, ‘one fine dog’, etc. There, it functions just like the single indefinite article ‘a(n)’. Dictionaries often offer them as synonyms. For example, the *Webster's Revised Unabridged Dictionary*, (1998) gives as first definition of the word ‘one’: “The same word as the indefinite article ‘a’, ‘an’.” Interestingly enough, the *American Heritage Dictionary of the English Language* (1996) conceives this use of ‘one’ not as an article, but as an adjective. However, that is mistaken, because ‘one’ will not replace most other adjectives. It is only substitutable

by the indefinite singular article a(n). Consider the aforementioned examples: ‘I’m just one player on the team’ is as correct as ‘I’m just a player on the team’ and ‘That is a fine dog’ is as grammatical as ‘That is one fine dog’. Nonetheless, this last example clearly shows that the role of ‘one’ in the nominal expression ‘one fine dog’ is completely different than that of the adjective ‘fine’. For one thing, without the adjective ‘fine’, the nominal expression does not change its grammatical status and the statement does not lose its grammatical correctness. Even though saying ‘That is one dog’ makes sense, saying ‘That is fine dog’ does not.³

Its singular article status prefixes it to a singular common noun (the sign of a concept or pseudo-concept) to create a nominal expression (the name of an object or a pseudo-object). For example, adding the word ‘one’ to the singular common noun ‘sailor in town’ results in the singular nominal expression ‘one sailor in town’. So far, it shares its grammatical role with other singular articles such as ‘the’, ‘a’ or ‘this’. However, since ‘one’ is an indefinite singular article, the resulting single nominal expression is an indefinite nominal expression. It designates a singular but indefinite object (or pseudo-object). The expression ‘the sailor in town’ is a definite name, while ‘one sailor in town’ is indefinite. In this sense, only the articles ‘a’ and ‘an’ can replace it, depending on the morphology (orthography) of the following word.

Nevertheless, when the word ‘one’ appears at the beginning of a singular nominal expression, sometimes it function as an adjective. Frequently, when the word ‘one’ expresses being the only individual of a specified or implied kind, it functions as an adjective. For example, in the complex nominal expressions ‘The one person I could marry’ or ‘The one horse that can win this race’, ‘one’ plays the role of an adjective. In these rare cases, it can

³. Frege uses a similar argument to show that, since ‘one’ is not an adjective, unity is not a property of objects.

replace the synonymous adjective ‘only’. The resulting expressions have the same grammatical status, such as ‘The only person I could marry’ or ‘The only horse that can win this race’. Notice, however, that the adjectival use of ‘one’ cannot replace every adjectival occurrence of ‘only’.⁴ For example, the sentence ‘Robert Smithson was an only child’ makes sense, while ‘Robert Smithson was an one child’ does not. ‘Robert Smithson was one child’ is also grammatically correct, where ‘one’ plays the role of the article, not of an adjective. Furthermore, the adjective ‘only’ modifies plural as well as singular nouns. ‘One’ functions as an adjective also when used synonymously with ‘united’ or ‘undivided’ as in “The church is therefore one, though the members may be many” or, most uncommon, synonymously with ‘single’ or ‘unmarried’, as in “Men may counsel a woman to be one.” The same stipulations apply to these cases as to ‘only’, except that, as adjectives, they can also complement the verb ‘to be’ to form a predicate. In summary, the category of ‘one’ as an adjective, is the disjunction of the category ADJECTIVE and the disjunction of the categories [‘only’], [‘united’] and [‘single’].

The second most common use of ‘one’ in ordinary English is that of a pronoun. In those cases, it expresses either an indefinitely specified individual, as in ‘She visited one of her cousins’, or an unspecified individual, as in ‘The older one grows the more one likes indecency’. In this later case, the indefinite pronoun ‘anyone’ can replace it. Being a pronoun means functioning alone as a nominal expression. Any nominal expression, including another pronoun of the same type, can replace it. This rule justifies the grammatical correctness of statements such as ‘The older anyone grows the more one likes indecency’ and ‘The older Virginia Woolf grows the more she likes indecency’. Like a

4. Furthermore, the word ‘only’ can also play an adverbial role, which is not true of ‘one’. For example, saying ‘I lived only for your love’ makes sense, but not ‘I lived one for your love’.

proper name, the word ‘one’ can stand alone as a whole name. Unlike the adjective ‘one’, the pronoun ‘one’ has a plural form. The pronoun ‘ones’ is the plural of pronoun ‘one’.

A third use of ‘one’ is that of a noun. For the most part, the noun ‘one’ is a numeral, naming number one. In those cases, it is part of arithmetical statements: not part of the external application of arithmetic, but of its internal application. ‘One’ is also a non-numerical common noun referring to a single person or thing: a unit, for example, ‘This is the one I like best’. In these cases, it is a common pronoun. Just as a proper pronoun may replace a noun or nominal expression, the word ‘one’ stands in place of a common noun. The word ‘one’, for example, may replace the common noun ‘dog’ in ‘This is the dog that bit me’ resulting in the grammatically correct sentence ‘This is the one that bit me’. It may even replace the whole complex common nominal expression ‘dog that bit me’ resulting in the sentence ‘This is the one’. This substitution requires a singular definite nominal expression containing the common noun. A singular definite pronoun such as ‘the’, ‘this’ or ‘that’ must precede it in a grammatically correct sentence.

If $P(x \ y)$ is a well formed sentence such that $(x \ y)$ is a nominal expression and x is a definite article, then $P(x \ 'one')$ is also a well formed sentence. In consequence, this category is $y [P(x \ y) \ W \ \& \ x \ SDA]$, where W is the category of well formed sentences and SDA is the category of singular definite articles. If the common noun occurs in a singular indefinite pronoun, and the indefinite article preceding the noun itself is ‘a’, ‘an’ or ‘one’, the rule does not apply. Articles ‘a’, ‘an’ or ‘one’ cannot precede the noun ‘one’, because the expressions ‘an one’, ‘a one’ and ‘one one’ do not make sense. If the common noun is part of a singular indefinite pronoun, and the indefinite article preceding the noun is ‘any’ or ‘some’, then ‘one’ fuses with them into the word ‘anyone’ or ‘someone’. Just as with the pronominal use of ‘one’, the common pronoun ‘ones’ is the plural of ‘one’.

‘Ones’ stands for common nouns in plural definite nominal expressions. As the grammatical correctness of ‘This one bit me’ derives from the correctness of ‘This dog bit me’, the correctness of ‘These dogs bit me’ entails the correctness of ‘These ones bit me’.

Finally, *The American Heritage Dictionary of the English Language* includes the following usage note for ‘one’:

Usage Note: When constructions headed by ‘one’ appear as the subject of a sentence or relative clause, there may be a question as to whether the verb should be singular or plural. Such a construction is exemplified in the sentence “One of every ten rotors was found defective.” Although the plural were is sometimes used in such sentences, an earlier survey found that the singular was preferred by a large majority of the Usage Panel. Another problem is raised by constructions such as ‘one of those people who’ or its variants. In the sentence “The defeat turned out to be one of the most costly blows that were ever inflicted on our forces,” most grammarians would hold that the plural were is correct, inasmuch as the subject of the verb is the plural noun ‘blows’. However, constructions of this sort are often used with a singular verb even by the best writers. Note also that when the phrase containing one is introduced by the definite article, the verb in the relative clause must be singular: “He is the only one of the students who has (not have) already taken.”⁵

Taking these uses into consideration,⁶ it is possible to define the number one as the grammatical category to which the English expression ‘one’ belongs. This category is the intersection of the five uses considered before. Since two of its synonyms – ‘only’ and ‘a’ – have no grammatical categories in common, this intersection captures the specificity of category [‘one’]. This is sufficient to define this unique category as the base of the system of cardinal numbers.

5. *The American Heritage® Dictionary of the English Language*, Third Edition. Houghton Mifflin Company, 1996, 1992.

6. English has a final, but obscure, use of the word ‘one’. ‘One’ is also a transitive verb meaning ‘to cause’ ‘to become one’; ‘to gather into a single whole’; ‘to unite’; ‘to assimilate’. For example: “The rich folk that embraced and oned all their heart to treasure of the world” [Chaucer] *Webster’s Revised Unabridged Dictionary*, © 1996, 1998 MICRA, Inc.

2. 'Two'.

The English word 'two' has less diverse uses than 'one'. It functions as noun, article and adjective. As expected, these uses of 'two' are analogous to those of 'one'. Just as 'one', the most common use of 'two' is in the composition of nominal expressions such as 'two friends of mine', 'two fine dogs', etc. Substituting 'one' for 'two' in a singular nominal expression and changing the number of the common nominal expression from singular to plural produces an analogous plural nominal expression. For example, the substitution of 'one' for 'two' and the 'chair' for 'chairs' in 'one broken chair' results in 'two broken chairs'. This does not mean that one nominal expression freely substitutes for the other. Saying 'I'm just one player on the team' makes sense, but 'I'm just two players on the team' does not. Singular and plural nominal expressions remain different grammatical categories. In the majority of these cases, 'two' works as a plural indefinite article for exactly two individuals. Placing 'two' before a common noun or nominal expression in the plural form results in a nominal expression. For example, preposing the common noun 'sailors in town' in its plural with the word 'two' produces the plural nominal expression 'two sailors in town'. As 'one', the word 'two' also occurs in nominal expressions as an adjective. As the word 'one' expresses 'being the only individual of a specified or implied kind', the word 'two' expresses 'being the two individuals of a specified or implied kind'. The complex nominal expressions 'The two persons I could marry' and 'The two scientists that synthesized the protein' are examples of this. When adjective 'one' is synonymous with 'only', substituting 'two' for 'one' and switching the common nouns and subordinate nominal clauses from singular to plural results in a grammatical correct plural nominal expression. This way, 'The one person who lives in this house' transforms into 'The two persons who live in this house'. However, this transformation does not apply to the other adjectival uses of 'one'. 'Two' cannot replace 'one', when meaning 'united' or 'single'.

Most of the times, noun ‘two’ expresses number two. Just as numeral ‘one’, the use of numeral ‘two’ in arithmetical statements is part of arithmetic’ internal *Anwendung*. Noun ‘two’ also refers to exactly two persons or things, for example in ‘These are my favorite two’.⁷ In these cases, it behaves like a common pronoun. Just as the noun ‘one’ may replace a singular common noun, ‘two’ may replace a plural common noun. For example, substituting the common noun ‘dogs’ in ‘These are the dogs that bit me’ for ‘two’ results in the grammatically correct sentence ‘These are the two that bit me’. ‘Two’ may even replace the whole complex nominal expression ‘dogs that bit me’ resulting in the sentence ‘These are the two’.⁸ This substitution is permissible when the common noun is part of a plural nominal expression whose main article is not ‘two’. In other words, it may occur preceded either by a plural definite pronoun such as ‘the’, ‘these’ or ‘those’, or by an indefinite one such as ‘any’ or ‘some’. If $P(x \ y)$ is a well-formed sentence such that $(x \ y)$ is a nominal expression and x is a singular article different from ‘two’, then $P(x \ \text{‘two’})$ is also a well-formed sentence. In consequence, the definition of this category is $y [P(x \ y) \ W \ \& \ x \ SA \ \& \ x \ \text{‘two’}]$, where W is the category of well-formed sentences and SA is the category of singular definite articles.

c. From ‘one’ to ‘two’ and beyond: succession and induction.

An important feature results from analyzing the grammar of the English words ‘one’ and ‘two’. The major uses of these words are analogous, requiring only minor adjustments between singular and plural⁹. Once made the proper number arrangements, and except for

7. As a matter of fact, another nominal use of ‘two’ exists. ‘Two’ also names something with two parts, units, or members, especially a playing card, the face of a die, or a domino with two pips.

8. Notice that in the sentence ‘these are two’, ‘two’ behaves like an adjective.

9. ‘One’ easily substitutes ‘two’, but not vice versa.

the cases where ‘two’ functions as a numeral, ‘one’ may replace ‘two’ any time it occurs in English. Furthermore, ‘two’ and the rest of the cardinal numerals, such as ‘three’, ‘five’ or ‘five hundred’ share these three basic uses: (i) plural indefinite article – as in ‘I ate three glazed doughnuts yesterday’, (ii) adjective – as in ‘These are my favorite ten jazz albums of all time’, and (iii) indefinite pronoun – as in ‘I’ll take six, please’. Nevertheless, to define the category of cardinal number, it is necessary to make sure that no other word have these same grammatical uses.

Cardinal numerals share their grammatical role of plural indeterminate articles with words such as ‘some’ or ‘many’¹⁰ (but not with plural determinate articles such as ‘these’, ‘those’ and ‘the’). Consider the context $\neg x$ (‘I know (that) $C(x, y)$, but cannot tell which’) where $C(x, y)$ is a well-formed sentence and y is a plural common noun. It determines the category of indeterminate article. ‘I know that five men in this island are married, but I cannot tell which’ makes sense, despite the nonsense of ‘I know that the men in this island are married, but I cannot tell which’ or ‘I know that those men in this island are married, but I cannot tell which’. However, this context does not define the category of cardinal numbers. ‘I know that some men in this island are married, but I cannot tell which’ and ‘I know that many men in this island are married, but I cannot tell which’ both make sense. Wittgenstein addresses this problem in Appendix 8 of the first part of the *Philosophical Grammar*, entitled “the concept ‘about’. The problem of the heap’ [*Der Begriff “ungefähr”. Problem des ‘Sandhaufens’*].

In that Appendix, Wittgenstein defines the grammatical category of cardinal numbers through a context similar to the following: $\neg x$ ‘there aren’t x apples on the table anymore, for I took y ’. If ‘one’ substitutes for y , only cardinal numerals for numbers larger

¹⁰. In PG PT. II Section IV §18, Wittgenstein writes: “Es gibt auch ein Zahlensystem ‘1, 2, 3, 4, 5, viele’.” “There is also a system of numbers ‘1, 2, 3, 4, 5, many’.”

than one may replace x . The context x ‘there aren’t x apples on the table anymore, for I took one’ defines the category \sim of cardinal number (larger than one). In general, the substitution is grammatically correct only if $x > y$. This already allows us to define ‘two’ and, in general, the rest of the cardinal numbers through their use as indeterminate articles. For example, the contexts x ‘there aren’t three apples on the table anymore, for I took x of them’ and x ‘I took x apples from the three on the table’ determine the category two = [‘two’]. Furthermore, the grammatical definition of the successor function is:

$$S(n) = x \text{ ‘there aren’t } x \text{ apples on the table anymore, for I took } n \text{ of them’ } y \\ \text{ [‘there aren’t } y \text{ apples on the table anymore, for I took } n \text{’ } (\text{‘there aren’t } y \text{ apples} \\ \text{ on the table anymore, for I took } x \text{’ } v \sim x).$$

Including the following induction principle, would conclude the grammatical definition of numerical system:

$$C [C(\text{‘one’}) \ \& \ n \sim (C(n) \ C(Sn))] \ (C = \sim).$$

This principles says that every context such that ‘One’ can fill its blank spaces, and any numeral can replace its predecessor, determines the category of cardinal number. However, this principle is false. In strict sense, no other cardinal numeral can replace ‘one’ in any context. ‘One’ is singular, and the rest of the numerals are plural. Different interpretations of this grammatical fact exist. It is possible to say that ‘one’ is not a cardinal number in strict sense. However, some simple adjustments of number allow for ‘two’ to succeed ‘one’. Still ‘two’ may not replace ‘one’ in all the contexts, even making the necessary adjustments of number. For example, consider the context ‘The plate smashed in n pieces’.¹¹ Saying ‘The plate smashed in two pieces’ makes sense, but ‘The plate smashed in one piece’ does not. It would seem that ‘two’ is not a cardinal either. Similarly, only numerals above two satisfy

^{11.} x (‘This rectangle consists of x parts’) PG Pt. II section IV 618 p. 638 (p. 324).

the context n ('This figure has n sides'). It makes sense to say 'This figure has three sides', but not to say 'This figure has two sides'.

Es hat keinen Sinn, von einem schwarzen Zweieck in weißen Kreis zu reden; und dieser Fall ist analog dem: es ist sinnlos zu sagen, das Viereck bestehe aus = Teilen (keinem Teil). Hier haben wir etwas, wie eine untere Grenze des Zählens, noch ehe wir die Eins erreichen. [PG PT. II Section IV §18 p. 640]

It makes no sense to speak of a black two-sided figure in a white circle; this is analogous to its being senseless to say that the rectangle consists of 0 parts (no part). Here we have like a lower limit of counting before we reach the number one. [PG Pt. II Section IV § 18 p. 326]

Und hier entsteht nun der Irrtum: Man meint, da man von der Zwei- und Vier-Teilung sprechen kann, kann man auch von der Zwe Drei-Teilung sprechen, geradeso, wie man 2, 3 und 4 Äpfel zählen kann. Aber die Drei-Teilung – wenn es sie guabe – gehört ja einer ganz anderen kategorie, einem ganz anderen System an als die Zwei- und Vier-Teilung. In dem System, in dem ich von Zwe- und Vier-Teilung sprechen kann, kann ich nicht von Drei-Teilung sprechen. Das sind logisch ganz verschiedene Gebilde. [PR Appendix II p. 321]

And this is where the mistake occurs: people think, since we can talk of dividing into 2, into 4 parts, we can also talk of dividing into 3 parts, just as we can count 2, 3 and 4 apples. But trisection – if there were such a thing – would in fact belong to a completely different system, from bisection, quadrisection. In the system in which I talk of dividing into 2 and 4 parts I can't talk of dividing into 3 parts. These are completely different logical structures. [PR Appendix II p. 334]

For Wittgenstein, this means that cardinal numbers have no unique single system. Instead a non-hierarchical motley of cardinal, numerical systems exists. Each countable concept determines its own numerical system with its own base. Apples start in one, parts in two, and sides of a geometrical figure in three. No system of cardinal numbers is primary. In consequence, asking whether the induction principle holds for the cardinal numbers does not make sense. The induction principle is not a result of Wittgenstein's grammatical analysis. However, this is not a flaw or weakness. The question is not whether the induction principle is true or false, but for what numerical systems it holds.

B. Calculation: the case of division.

Basic arithmetical equations result from the grammatical analysis of natural language. Consider the case of division among natural numbers as an example of arithmetical calculation.¹² The calculation ‘ $11 \div 3 = 3$ ’ provides the grammar of the statement ‘if I have eleven apples and want to share them among some people so that each is given three apples, I can give three people their share with two apples remaining’. In general, ‘if I have x apples and want to share them among some people so that each is given y apples, I can give z people their share with w apples remaining’ is grammatically correct if z is the result of dividing x between y , and w is the remanent. Hence, x, y, z, w ‘if I have x apples and want to share them among some people so that each is given y apples, I can give z people their share with w apples remaining’ grammatically defines the four-place relation that z is the result of dividing x between y , and w is the remanent. In consequence, the context x, y, z, w (‘if I have x apples and want to share them among some people so that each is given y apples, I can give z people their share with w apples remaining’) in natural language determines the same category as the context

$$x, y, z, w \left(\frac{w}{y / x} \right)$$

in the internal *Anwendung* of arithmetic.

Every calculation statement in the calculus provides the grammar for some statement in natural language. The abstraction of different numerical expressions from that sentence determine categories co-extensional with the calculation and result concepts in the original equation. In general, every grammatical context in the calculus has an analogous in natural

¹². The example is not fortuitous. Wittgenstein illustrate the notion of external *Anwendung* with addition in §15 of section III of the second part of the *Philosophical Grammar*. The previous chapter analyzed this example in detail.

language. Both contexts determine the same category. This correspondence justifies the reproduction of the results of the grammatical analysis in section II for the external *Anwendung* of arithmetic.

IV. Conclusion: A False Dilemma

Interpretations of the thesis that mathematical propositions are grammatical diverge in one major respect. Wittgenstein scholars agree that mathematical propositions are grammatical rules for the construction or transformation of statements. However, they disagree on whether or not they are rules of mathematical or natural language. The basis of this disagreement is a false dilemma. Both interpretations are correct, but incomplete when considered in isolation.

It is tempting to approach Wittgenstein's idea that mathematics is grammar as an analogy. For Wittgenstein, images, metaphors and analogies are important sources of philosophical insight and confusion. However, Wittgenstein does not use 'grammar' metaphorically. Mathematical propositions are not *like* grammatical rules. They *are* grammatical rules. Calculi are languages and calculations are expressions. The propositions of a calculus constitute its syntax, because they determine the calculations correctness. Calculation is the construction of linguistic expressions in accordance with the calculus grammar. Correct calculations are correct expressions in the calculus language. For example, the axioms and theorems of arithmetic constitute the syntax of arithmetic calculation. Similarly, the axioms of geometry are the syntactic rules of geometrical construction. In general, calculations within a mathematical theory are linguistic constructions in a language, with the theory's axioms as the rules of its syntax.

Mathematics is also part of the grammar of natural language. This chapter has proved that arithmetic is the syntax of numerical expressions in natural language. The

application of the proper grammatical analysis to cardinal numerical expressions in natural language results in a grammar with a natural arithmetical interpretation. It proved that cardinal numerical expressions belong to a unique grammatical category, and that the basic axioms of arithmetic determine their grammar.

For Wittgenstein, looking for an unique universal grammar underlying the whole of natural language and its uses is the biggest mistake of conventional grammarians. They ignore the inherent and essential multiplicity of language. Instead of a unique grammar for the whole of language, grammarians ought to look for the many grammars behind the many uses of language. Then, they will realize that some of these grammars are some of the most successful mathematical calculi. Euclidean geometry, for example, is the grammar of natural language *when used for the description of objects in the visual space*. Every mathematical calculus, not only arithmetic, constitute a portion of natural language grammar. For example, the axioms of Euclidean geometry comprise the syntax of natural language descriptions of objects in visual space.

Conceiving mathematics only as part of the syntax of natural language is also a mistake. For Wittgenstein, mathematical calculi regulate both a variety of mathematical activities, such as counting, measuring, adding, etc. and a part of the syntax of natural language. The dilemma between these two interpretations vanishes with the realization that mathematical language *is* a part of natural language. Mathematical calculi do not need to be performed in an artificial symbolic or diagrammatical language. They can be expressed in terms of apples, pebbles or whatever. This way they are incorporated into natural language. Arithmetic, like the rest of mathematics, is a subsystem inside natural language. Statements about adding apples or subtracting pebbles *is* as much arithmetic as statements about adding or subtracting numbers. This segment of natural language obeys mathematics, because it constitutes its grammar.

Furthermore, both dimensions of grammar are identical. For example, measuring is calculation because it is a technique for the construction of linguistic expressions. One does not exist without the other. As a linguistic technique, measuring obeys a syntax and, as a calculation, it obeys the rules of a calculus. Both systems of rules are identical. The syntax of calculation is always ultimately mathematical.

The results of this fifth chapter are twofold. On the one hand, it has formally shown that numerical calculi actually constitute grammatical systems in the sense of Wittgenstein. On the other hand, it also showed that they are the grammar of their segment of natural language. It has proved that if the object language contains the appropriate numerical expressions, the resulting grammar will include at least some rules with a natural mathematical interpretation. A grammatical analysis of using numerical expressions, both in calculation and in natural language, has yielded familiar theorems of arithmetic.

Chapter 7

Zahlangaben Revisited

I. Introduction

This chapter fits Wittgenstein's account of grammatical proposition within the framework of his philosophy of mathematics. Chapters 2 and 3 developed Wittgenstein's theory of mathematical propositions and calculation, while chapters 4, 5 and 6 worked on the formal demonstration of the grammatical nature of mathematical propositions. This chapter brings the result of this latter research to answer the questions about *Zahlangaben* left unanswered in the first two chapters. The first section offers a new formal approach to the notion of grammatical concept and relates it to that of 'system of propositions'. It also finishes the grammatical account of mathematical *Zahlangaben* started in Chapter 1. The philosophical and formal developments of the previous chapter allow for the completion of this explanation. The sum of these results substantiates Wittgenstein's understanding of mathematical *Zahlangaben* as grammatical propositions

II. Grammatical Concepts and Systems of Propositions

A. Systems of Propositions

At the completion of the *Tractatus*, Wittgenstein had isolated himself from academic philosophy, except for some conversation and correspondence with Ramsey¹ and a few members of the Vienna Circle. Ramsey's criticisms of his logical atomism partially inspired Wittgen-

1. Wittgenstein's conversation and correspondence with the mathematician Frank Ramsey, whom he befriended during the twenties and until Ramsey's premature death at the age of twenty six in January 1930, included many of Wittgenstein's post-*Tractatus* ideas on mathematics. Ramsey praised Wittgenstein's extensive work on the philosophy of mathematics during this period. Hans Sluga, introduction to *The Cambridge Companion to Wittgenstein*. Vol. 3, *From the Tractatus to Remarks on the Foundations of Mathematics: Wittgenstein on the philosophy of mathematics* (New York: Cambridge University Press, 1996) 18.

stein's return to professional philosophy. Wittgenstein's grammatical approach during this middle period is his response to this criticism, as the following passage, from F. Waismann's notes from Christmas, 1929, confirms:

Ich habe einmal geschrieben: 'Der Satz ist wie ein Maßstab an die Wirklichkeit angelegt. Nur die äußersten Teilpunkte berühren den zu messenden Gegenstand'. Ich möchte jetzt lieber sagen: Ein 'Satzsystem' ist wie ein Maßstab an die Wirklichkeit angelegt. Ich meine damit folgendes: Wenn ich einen Maßstab an einem räumlichen Gegenstand anlege, so lege ich *alle Teilstriche* zu gleicher Zeit an. Nicht die einzelnen Teilstriche werden angelegt, sondern die ganze Skala. Weiß ich, daß der Gegenstand bis zum Teilstrich 10 reicht, so weiß ich auch unmittelbar, daß er nicht bis zum Teilstrich 11, 12 usw. reicht. Die Aussagen, welche mir die Länge eines Gegenstandes beschreiben, bilden ein System, ein Satzsystem. Ein solches ganzes Satzsystem wird mit der Wirklichkeit verglichen, nicht ein einzelner Satz. Wenn ich z. B. sage, der und der Punkt im Gesichtsfeld ist *blau*, so weiß ich nicht nur das, sondern auch, daß der Punkt nicht grün, nicht rot, nicht gelb usw. ist. Ich habe die ganze *Farbenskala* auf einmal angelegt. Das ist auch der Grund dafür, warum ein Punkt zu gleicher Zeit nicht verschiedene Farben haben kann; warum ein syntaktisches Verbot besteht, daß fx für mehr als einen Wert x wahr sein kann. Denn wenn ich ein Satzsystem an die Wirklichkeit anlege, so ist damit – genau wie beim räumlichen – schon gesagt, daß immer nur *ein* Sachverhalt bestehen kann, nie mehrere.

Ich habe all das bei der Abfassung meiner Arbeit noch nicht gewußt und meinte damals, daß alles Schließen auf den Form der Tautologie beruhe. Ich hatte damals nicht gesehen, daß ein Schluß auch die Form haben kann: Ein Mensch ist 2 m groß, also ist er nicht 3 m groß. Das hängt damit zusammen, daß ich glaubte, die Elementarsätze müßten unabhängig sein; aus dem Bestehen eines Sachverhaltes könne man nicht auf das Nichtbestehen eines andern Schließen. Wenn aber meine jetzige Auffassung mit dem Satzsystem richtig ist, so ist sogar die Regel, daß man aus dem Bestehen eines Sachverhaltes auf das Nichtbestehen aller übrigen schließen kann, die durch das Satzsystem beschrieben werden. [PR Appendix II, p. 304]

I once wrote: 'A proposition is laid like a yardstick against reality. Only the outermost tips of the graduation marks touch the object to be measured'. I should now prefer to say: a *system of propositions* is laid like a yardstick against reality. What I mean by this is: when I lay a yardstick against a spatial object, I apply *all the graduation marks simultaneously*. It's not the individual graduation marks that are applied, it's the whole scale. If I know that the object reaches up to the tenth graduation mark, I also know immediately that it doesn't reach the eleventh, twelfth, etc. the assertions telling me the length of an object form a system, a system of propositions. It's such a whole system which is compared with reality, not a single proposition. If, for instance, I say such and such a point in the visual field is *blue*, I not

only know that, I also know that the point isn't green, isn't red isn't yellow etc. I have simultaneously applied the whole color scale. This is also the reason why a point can't have different colors simultaneously; why there is a syntactical rule against fx being true for more than one value of x . For if I apply a *system* of propositions to reality, that of itself already implies – as in the spatial case – that in every case only *one* state of affairs can obtain, never several.

When I was working on my book I was still unaware of all this and thought then that every inference depended on the form of a tautology. I hadn't seen then that an inference can also be of the form: A man is 6 ft tall, therefore he isn't 7 ft. This is bound up with my then believing that elementary propositions had to be independent of one another: from the fact that one state of affairs obtained you couldn't infer another did not. But if my present conception of a system of propositions is right, then it's a rule that from the fact that one state of affairs obtains we can infer that all the others described by the system of propositions do not. [PR Appendix II p. 317]

In the second paragraph of this passage, Wittgenstein refers to the *Tractatus Logico-Philosophicus* as 'My book'. In 2.1512, he wrote that a proposition lays against reality like a ruler [*Es ist wie ein Maßstab an die Wirklichkeit angelegt*]. The passage also refers to 2.15121, where he added "Only the end points of the marks on the ruler touch the object being measured" [*Nur die äußeren Punkte der Teilstriche berühren den zu messenden Gegenstand*]. For the Wittgenstein of the early thirties, systems of propositions, not single propositions, work as rulers to measure reality.

Die Sätze werden in diesem Falle noch ähnlicher Maßstäben, als ich früher geglaubt habe. - Das Stimmen eines Maßes schließt automatisch alle anderen aus. Ich sage automatisch: Wie alle Teilstriche auf einem Stab sind, so gehören die Sätze, die den Teilstrichen entsprechen, zusammen, und man kann nicht mit einem von ihnen messen, ohne zugleich mit allen andern zu messen. - Ich lege nicht den Satz als Maßstab an die Wirklichkeit an, sondern das System von Sätzen.

Man könnte nun die Regel aufstellen, daß derselbe Maßstab in einem Satz nur einmal angelegt werden darf. Oder, daß die Teile, die verschiedenen Applikationen desselben Maßstabes entsprechen, zusammengefaßt werden müssen. [PR §82 p. 100]

In which case, propositions turn out to be even more like yardsticks than I previously believed. – The fact that *one* measurement is right automatically excludes all others. I say automatically: just as all the graduation marks are on *one* rod, the propositions corresponding to the graduation marks

similarly belong together, and we can't measure with one of them without simultaneously measuring with all the others. – It isn't a proposition which I put against reality as a yardstick, it's a *system* of propositions.

We could now lay down the rule that the same yardstick may only be applied once in one proposition. Or that the parts corresponding to different applications of one yardstick should be collated. [PR 82 p. 110]

Unsere Erkenntnis ist eben, daß wir es mit Maßstäben und nicht quasi mit isolierten Teilstrichen zu tun haben.

Jede Aussage bestünde dann gleichsam im Einstellen einer Anzahl von Maßstäben, und das Einstellen eines Maßstabes auf zwei Teilstriche zugleich ist *unmöglich*. [PR §84 p. 112]

What we have recognized is simply that we are dealing with yardsticks, and not in some fashion with isolated graduation marks.

In that case every assertion would consist, as it were, in setting a number of scales (yardsticks) and it's *impossible* to set one scale simultaneously at two graduation marks. [PR §84 p. 112]

Wittgenstein developed the transition from elementary propositions to a system of propositions in more depth in section VIII of the *Philosophical Remarks*, where he writes:

§83 Der Begriff des "Elementarsatzes" verliert jetzt überhaupt seine frühere Bedeutung.

Die Regeln über "und", "oder", "nicht" etc., die ich durch die W-F-Notation dargestellt habe, sind ein Teil der Grammatik über diese Wörter, aber nicht die ganze.

Der Begriff der unabhängigen Koordinaten der Beschreibung: Die Sätze, die z.B. durch "und" verbunden werden, sind nicht voneinander unabhängig, sondern sie bilden ein Bild und lassen sich auf ihre Vereinbarkeit oder Unvereinbarkeit prüfen.

In meiner alten Auffassung der Elementarsätze gab es keine Bestimmung des Wertes einer Koordinate; obwohl meine Bemerkung, daß ein farbiger Körper in einem Farbenraum ist etc., mich direkt hätte dahin bringen können.

Eine Koordinate der Wirklichkeit darf nur einmal bestimmt werden. [PR §83 p. 101]

§83. The concept of an 'elementary proposition' now loses generally its earlier significance.

The rules for 'and', 'or', 'not' etc., which I represented by means of the T-F notation are *a part* of the grammar of these words, but not *the whole*.

The concept of independent coordinates of description: the propositions joined, e. g., by 'and' are not independent of one another, they form *one* picture and can be tested for their compatibility or incompatibility.

In my old conception of an elementary proposition there was no determination of the value of a co-ordinate; although my remark that a coloured body is in a colour-space, etc., should have put me straight on to this.

A co-ordinate of reality may only be determined *once*.

If I wanted to represent the general standpoint I would say: ‘You should not say now one thing and now another about the same matter’. Where the matter in question would be the coordinate to which I can give *one* value and no more. [PR §83 p. 111]

Instead of atomic and molecular propositions, reality is described through sets of coordinates. Ascribing an attribute to an object is fixing a coordinate of reality. Hence, it can only be determined *once*. Even *zero* or *nothing* remain a value. In the *Tractatus*, Wittgenstein considered each ‘elementary proposition’ as one single coordinate which could only take two possible values, *true* or *false*. He also considered elementary propositions logically independent from each other. By the early thirties, Wittgenstein had recognized the error in his previous view. In the middle period, Wittgenstein wanted to construe a complete logical space where propositions are like points in a geometrical space, related through geometric/grammatical laws. Logical space remains for the old logic, because *true* and *false* are still suitable values for certain coordinates. However, other coordinates need a new *logic*.

1. Systems of Propositions Formally Defined

A system of propositions is a set of propositions in the form Cx associated with a context x (Cx). However, not all contexts yield systems of propositions. A system of propositions must also satisfy the further condition that one and only one of its propositions is true.

Wie verhält es sich aber mit allen scheinbar ähnlichen Aussagen, wie: Ein materieller Punkt kann nur *eine* Geschwindigkeit auf einmal haben, in eine Punkt einer geladenen Oberfläche kann nur *eine* Spannung sein, in einem Punkt einer warmer Fläche nur *eine* Temperatur zu *einer* Zeit, in *einem* Punkt eines Dampfkessels nur *ein* Druck etc.? Niemand kann daran

zweifeln, daß das alles Selbstverständlichkeit sind und die gegenteiligen Aussagen Widersprüche. [PR §81, p. 99]

What about all assertions which appear to be similar, such as: a point mass can only have *one* velocity at a time, there can only be *one* charge at a point of an electrical field, at one point of a warm surface only *one* temperature at one time, at one point in a boiler only *one* pressure etc.? No one can doubt that these are all self-evident and that their denials are contradictions. [PR §81 p. 109]

Definition 5.1.1 [system of propositions]: Given a grammatical categories G and a context $x(C(x))$ in the language L , the set of expressions resulting from the substitution of expressions in G in context C , $S = \{ C(e) \mid e \in G \}$ is a *system of propositions* iff $C \in G$ and $\neg C(e) \in S$ which is true.

The propositions ‘This page is 1 in. wide’, ‘This page is 2 in. wide’, ‘This page is 3 in. wide’, ‘This page is 4 in. wide’, etc. form a system of propositions associated with the category of the width of this page, because one and only one of them obtains, in such way that,

- (i) even though only one of them obtains, they are all possible, they all make sense,
- (ii) that they are all false is impossible (saying ‘This page has no width’ does not make sense), and
- (iii) if one of them obtains, the others are false.

Notice that two grammatical categories associated are at play in every system of propositions, and they are not necessarily the same. In the previous example, the context $x(\text{‘This page is’}^{\wedge}x)$ does not yield the category ‘width of this page’. Consider, well-formed-sentences like ‘This page is white’ or ‘This page is my dissertation’s 165th’. Clearly, ‘white’ or ‘my dissertation’s 165th’ are not expressions of this page’s width. However, it is not necessary that the abstraction of one width expression produce the

context associated to the system of propositions. It is sufficient that the width expressions satisfy the relevant context. In definition 5.1.1, that G is a subset of C guarantees that every substitution of expressions of G in C yield a well-formed-sentence.

2. Complete Descriptions

Definition 5.1.2 [complete description]: Every proposition in a system of propositions is a *complete description*.

Wittgenstein calls the propositions in a system “complete descriptions”². They fully determine co-ordinates of reality by assigning them a value. The value assigned to an object’s velocity of at a certain moment completely describes that velocity. It says everything about the velocity of that object at that time.³

Wie ist es möglich, daß $f(a)$ und $f(b)$ einander widersprechen, wie es doch der Fall zu sein scheint? z.B., wenn ich sage, "hier ist jetzt rot" und "hier ist jetzt grün"?

Es hängt das mit der Idee der vollständigen Beschreibung zusammen: "Der Fleck ist grün", beschreibt den Fleck vollständig, und es ist für eine andere Farbe kein Platz mehr.

§77 How is it possible for $f(a)$ and $f(b)$ to contradict one another, as certainly seems to be the case? For instance, if I say ‘there is red here and now’ and ‘there is green here now’?

This is connected with the idea of a *complete description*: ‘The patch is green’ describes the patch completely, and there’s no room left for another color. [PR §77 p. 106]

2. Wittgenstein uses the expression ‘complete description’ to emphasize the contrast with the *Tractatus* notion of ‘complete analysis’. At that time, Wittgenstein believed that “a proposition isn’t an elementary proposition unless its complete logical analysis shows that it isn’t built out of other propositions by truth-functions.” [PG Appendix I 4B p. 211] By the thirties, he had realized that his previous notion of logical analysis was senseless, and, in consequence, changed his definition of elementary proposition to one that “doesn’t contain a truth-function and isn’t defined by an expression which contains one” [PG Appendix I §4B p. 211]. Cf. also *Notizbücher* 06.17.1915.

3. Cf. *Zettel* Sct 311 ln 6: “Eine Rede vollständig (oder unvollständig) wiedergeben. Gehört dazu auch die Wiedergabe des Tonfalls, des Mienenspiels, der Echtheit oder Unechtheit der Gefühle, der Absichten des Redners, der Anstrengung des Redens? Ob das oder jenes für uns zur vollständigen Beschreibung gehört, wird vom Zweck der Beschreibung abhängen, davon, was der Empfänger mit der Beschreibung anfängt”.

Zu sagen, daß eine bestimmte Farbe jetzt an einem Ort ist, heißt diesen Ort vollständig beschreiben. [PR §80 p. 98]

To say that a particular color is now in a place is to describe that place *completely*. [PR §80 p. 108]

Understanding these definitions requires the introduction of a more definite definition of grammatical concepts.

3. Grammatical Concepts Formally Defined

Chapter 1 presented grammatical concepts as the logical product of its members – an extended disjunction. However, Wittgenstein’s notion of ‘system of propositions’ allows for a more precise presentation of grammatical concepts, evidencing their grammatical nature.

Definition 5.2.1 [semantic counterpart]: Concept S is a *semantic counterpart* of grammatical category $C = \lambda x (Px)$ iff for every expression e , $S(e)$ iff $e \in C = \lambda x (Px)$. In other words, a concept S is the semantic counterpart of a grammatical category C iff it contains the meaning of those and only those expressions in category C .

Definition 5.2.2 [internal description]: Proposition $S(e)$ is an internal description iff S is the semantic counterpart to a grammatical category C such that ‘ e ’ belongs to C .

Definition 5.2.3 [grammatical concept]: Concept C is a grammatical concept if and only if it is a grammatical category or the semantic counterpart of one.

B. Grammatical Concepts and Systems of Propositions

These definitions clarify Wittgenstein’s examples in the previous passages from the *Philosophical Remarks*.. Consider the three examples Wittgenstein offers in the passage quoted above – length, height and color, and those in section VIII – velocity, temperature, electrical charge and pressure. They are all grammatical categories. The concept of velocity, for example, forms a system of propositions with the context resulting from abstracting the

particular value in any velocity statement. The same holds for pressure, electrical charge and temperature. Take a physical statement that says that a certain point in an object has a given temperature at a fixed moment in time, for example, ‘The tip of my pencil was at 21°C at the stroke of midnight on New Year’s Eve 2000’. Abstract the actual temperature predicated in the statement: 21°C. Let C be the resulting context, x (‘The tip of my pencil was at x at the stroke of midnight on New Year’s Eve 2000’). All acceptable substitutions of temperature expressions for x form the system of propositions: { ‘The tip of my pencil was at 5°K at the stroke of midnight on New Year’s Eve 2000’, ‘The tip of my pencil was at 0°C at the stroke of midnight on New Year’s Eve 2000’, ‘The tip of my pencil was at 10°C at the stroke of midnight on New Year’s Eve 2000’, . . .). Every proposition in the system expresses a possible state of affairs in the world. The temperature expressions that satisfy C – ‘0°C’, ‘10°C’, ‘5°K’, ‘100°K’, etc. – refer to different temperatures. Furthermore, that the tip of my pencil has one and only one temperature is syntactically necessary. Since temperature measures can be made in more than one scale (Kelvin, Centigrade, etc.), the substitution of a temperature expression for x in T resulting in a true proposition is not unique. However all the expressions that do express the same temperature.

Color is a grammatical concept, because if it makes sense to predicate a color from a point in the visual field at a certain moment, it must be of one and only one color. In other words, given any context resulting from the abstraction of a color term from a complete description, any replacement of the color term producing a true statement must refer to the same color. The same holds for height. Anything that has some height has one and only one height. The category ‘height’ forms a system of propositions with the context x (‘I

am' x 'tall'), because x only has one value. Every height expression that may replace x , resulting in a true statement, must express the same single height.⁴

This does not mean that optics, dynamics or thermodynamics may reduce to grammar, or that temperature, color and velocity are not actual physical concepts, but disguised linguistic ones. However, the scale used for measuring the value of these concepts is a grammatical convention indeed. The Centigrade and the Kelvin systems are yardsticks for the measurement of temperature. The scheme of colors for ascribing color to spots in visual space is also a grammatical yardstick.

Die Einheitsstrecke gehört zum Symbolismus. Sie gehört zur Projektionsmethode. Ihre Länge ist willkürlich . . . Wenn ich also eine Strecke "3" nenne, so bezeichnet hier die 3 mit Hilfe der im Symbolismus vorausgesetzten Einheitsstrecke.

Dasselbe kann man auch auf die Zeit anwenden. [PR §45 p. 69]

The unit length is part of the symbolism. It belongs to the method of projection. Its length is arbitrary. . . And so if I call a length '3', the 3 signifies via the unit length presupposed in the symbolism.

You can also apply these remarks to time. [PR §45 p. 79]

Predicating a grammatical concept is significantly different from assigning a value to a grammatical concept. A grammatical proposition predicates a grammatical concept. A genuine proposition assigns it a value. For example, saying that a point in visual space is colored is very different from saying which color it is. The first proposition is grammatical,

4. This idea committed Wittgenstein to claim that every calculation has one result and that mathematical propositions with different proofs are actually altogether different propositions. However, these consequences are not as absurd as they seem. Remember that Wittgenstein views calculations and proofs as rule-governed activities. That they have one single and unique result is not surprising. If a calculation did not arrive at any result, it would not be a calculation. Talking about calculations with more than one result is also nonsense. The obvious case of equations with more than one root is not a counterexample, because it actually hides a multiplicity of mathematical propositions. The different process arriving at each root is a different calculation. Furthermore, discovering that they are both roots of the equation requires yet another calculation. In either case, Wittgenstein's rule of syntactic necessity expresses the close connection between process and result.

while the latter is genuine. Whether or not a concept applies to a term is a grammatical issue. The concept's value is not. 'Color' is a grammatical concept. However, this does not mean that optics is nothing but grammar. This is true, not only of colors, but of other grammatical concepts. For example, determining whether a statement is a proposition is a grammatical calculation. Whether it is true or false is not.

1. A Warning Notice

It is important not to be too hasty at pairing categories with contexts to form systems of propositions. Consider the following example. It is possible that, at noon, on my twenty first birthday, I murdered Mrs. Von Bulow. In other words, the statement 'I murdered Mrs. Von Bulow' is acceptable in ordinary English. The grammatical category x ('At noon, on my twenty first birthday, I murdered') contains proper names like 'Mrs. Von Bulow', 'Nezahualcoyotl', 'John F. Kennedy', etc., pronouns like 'you', 'myself', etc. and other expressions referring to persons. It would seem that the concept 'person' forms a system of propositions with the grammatical context x ('At noon, on my twenty first birthday, I murdered'). For every expression e in x ('At noon, on my twenty first birthday, I murdered'), e 'is a person' is true: Mrs. Von Bulow was a person, you are a person, I am a person myself, etc. However, this is not so. Otherwise, the statement 'At noon, on my twenty first birthday, I murdered one and only one person' would be necessary. From the possibility of me murdering someone, it would follow of necessity that I murdered one and only one person. However, that I murdered one and only one person is certainly not necessary.

Suppose that, on my twenty-first birthday, I threw a bomb at a train. The bomb exploded killing all of its passengers, some of them at the same time, exactly at noon.⁵ This example shows that, at noon, on my twenty first birthday, I could have murdered more than one person. In that case, more than one expression in the category ‘person’ might have produces a true statement when substituted in the relevant context. If, by throwing that bomb at the train, I had killed Ana Mendieta, John F. Kennedy and Gustavo Colossio, then all the expressions ‘Ana Mendieta’, ‘John F. Kennedy and Gustavo Colossio’ and ‘Gustavo Colossio, Ana Mendieta and John F. Kennedy’ would belong to the category ‘person’ and would satisfy x (‘At noon, on my twenty first birthday, I murdered’ x) producing a true statement. In consequence, that I murdered one and only one person is certainly not necessary.

These two counterexamples show that the concept person does not form a system of propositions with grammatical category x (‘At noon, on my twenty first birthday, I murdered’ x).

III. *Zahlangaben* Revisited

Chapter 5 demonstrated the existence of a grammatical category in natural language corresponding to the mathematical concept of cardinal number. Genuine *Zahlangaben* form systems of propositions associated with this grammatical category. This category includes all cardinal numerical expressions, like ‘three’, ‘none’, ‘five hundred’, ‘three million’, etc. Since cardinal numbers express quantities, the concept ‘quantity’ is this category semantical counterpart. In consequence, *Zahlangaben* are complete cardinal descriptions, which form systems of propositions associated with ‘quantity’.

⁵. Thanks to Bo Ram Lee at Columbia University, for the example.

Consider a genuine *Zahlangabe*, like ‘There are three men on this island’. It expresses a possible state of affairs in the world. The abstraction of the actual quantity of men predicated in the statement results in the context x (‘There are x men on this island’). The system of expressions associated with ‘quantity’ and this context includes such propositions as ‘There are no men on this island’, ‘There are three men on this island’, and ‘There are three hundred men on this island’. In this system of propositions, the cardinal expressions depict the quantity of men on this island. Hence, there is only one quantity of men on this island.

Just as optics and thermodynamics do not reduce to grammar, quantities cannot reduce to grammar either. However, the system of cardinal numbers as a scale for measuring quantities is a grammatical convention. Just as with the Centigrade scale of temperature and the scheme of colors, the system of cardinal numbers is also a grammatical yardstick.

A. On the Extensionality of Mathematical Concepts

For Wittgenstein, mathematical concepts are not genuine concepts in the same sense as genuine concepts such as ‘spoon’, ‘chair’ or ‘person in this room’. Mathematical concepts are grammatical categories.⁶ Since they are actually grammatical categories, their extensions completely determine them. In a certain sense, they are their own extensions. Mathematical and logical concepts such as unity, permutation, function, et cetera are not genuine concepts, because they lack proper intensionality. Unlike genuine concepts, the intension and the extension of these pseudo-concepts are the same.

Ordinarily, the replacement of a function by a list (class) is mistaken. We say something different when we talk about a class given in extension and when we talk about a class given by a defining property. Intension and extension are not interchangeable. For example, it is not the same thing to say “I hate the man sitting in the chair” and “I hate Mr. Smith.” But it is

⁶ PR §116.

otherwise in mathematics. In mathematics there is no difference between “the roots of the equation $x^2 + 2x + 1 = 0$ ” and the list [1 | 1], or between “the number satisfying $x + 2 = 4$ ” and “2.” The roots, and 2, are not described in the way the person is who satisfies the description “the man sitting in the chair.” [WL *Philosophy for Mathematicians* 1932-33 §1 p. 206]

For Wittgenstein, this means that they are actually disjunctions of their elements.⁷ In paragraph 116 of the *Philosophical Remarks*, for example, he writes that “The permutations (without repetition) of AB are AB, BA. They are not the extension of a concept: they alone are the concept”.

Gehen wir nun zur Schreibweise “(x) · fx” über, so ist klar, daß dies eine Sublimierung der Ausdrucksform unserer Sprache ist: “es gibt Menschen auf dieser Insel”, “Es gibt Sterne, die wir nicht sehen”. Und einem Satz “(x) · fx” soll nun immer ein Satz “fa” entsprechen, und “a” soll ein Name sein. Man soll also sagen können: “(x) · fx, nämlich a und b” (“es gibt einen Wert von x, der fx befriedigt, nämlich a und b”), oder “(x) · fx, z. B., a”, etc. Und dies ist auch möglich in einem Falle wie: “es gibt Menschen auf dieser Insel, nämlich die Herrn A, B, C, D.” Aber ist es denn für den Sinn des Satzes “es gibt Menschen auf dieser Insel” wesentlich, daß wir sie benennen können, also ein bestimmtes Kriterium für die Identifizierung festlegen? Das ist es nur dann, wenn der Satz “(x) · fx” als eine Disjunktion von Sätzen von der Form “f(x)” definiert wird, wenn also z. B. festgelegt wird: “Es gibt Menschen in dieser Insel” heiße “Auf dieser Insel ist entweder Herr A oder B oder C oder D oder E”; wenn man also den begriff ‘Mensch’ als eine Extension bestimmt (was natürlich ganz gegen die normale Verwendung dieses Wortes wäre). (Dagegen bestimmt man z. B. Den Begriff “primäre Farbe” wirklich als eine Extension.) [PG Pt. I Appendix 2. p. 398]

If we turn to the form of expression “(x) · fx” it’s clear that this is a sublimation of the form of expression in our language: “There are human being on this island”, “There are stars that we do not see”. To every proposition of the form “(x) · fx” there is supposed to correspond a proposition “fa”,

7. Extensionality is a property of all grammatical specification concepts, not only mathematical ones. For example, regarding the concept ‘pure color’, which is a grammatical category according to Wittgenstein, he writes in PR §116: “If I am right, there is no concept ‘pure colour’; the proposition ‘A has a pure colour’ simply means ‘A is red, or yellow, or green, or blue’. ‘This hat belongs to either A or B or C’ isn’t the same proposition as ‘This hat belongs to someone in this room’, even if as a matter of fact only A, B and C are in the room, since that needs saying.” [Wenn ich recht habe, so gibt es keinen Begriff ‘reine Farbe’; der Satz ‘A hat eine reine Farbe’ heißt einfach ‘A ist rot, oder gelb, oder blau, oder grün’. ‘Dieser Hut gehört entweder A oder B oder C’ ist nicht derselbe Satz wie ‘dieser Hut gehört einem Menschen in diesem Zimmer’, selbst wenn tatsächlich nur A, B, C in Zimmer sind, denn das muß erst dazugesagt werden.]

and “a” is supposed to be a name. So one must be able to say “ $(x) \cdot fx$, namely a and b” (“There are some values of x , which satisfy fx , namely a and b”), or “ $(x) \cdot fx$, e. g. a”, etc. And this is indeed possible in a case like “There are human beings on this island, namely Messrs A, B, C, D.” But then is it essential to the sense of the sentence “There are men on this island” that we should be able to name them, and fix a particular criterion for their identification? That is only so in the case where the proposition “ $(x) \cdot fx$ ” is defined as a disjunction of propositions of the form “ $f(x)$ ”, if e. g. it is laid down that “there are men on this island” means “Either Mr. A or Mr. B or Mr. C or Mr. D or Mr. E is on this island” – if, that is, one determines the concept “man” extensionally (which of course is quite contrary to the normal use of this word.) (On the other hand the concept “primary colour” really is determined extensionally.) [PG Pt. I, Appendix 2. Pp. 203, 204]

Knowing a mathematical concept involves knowing its extension. They are not *indefinite*.⁸

For any grammatical category X , ‘ A is X ’ simply means ‘ A is x_1 , or x_1 , or x_1 , or . . .’ where x_1 ,

x_2, x_3, \dots are *the* X s⁹. Since grammatical categories are genuinely disjunctions, mathematical

propositions have no generality. Non-mathematical *Zahlangabe*, on the contrary, are general. By *general* [allgemeine], Wittgenstein means that their logical form is a quantified proposition. For instance, ‘3 men are in this room’ has the form ‘ $(x, y, z) : x$ is a man and is in this room and y is a man and is in this room and z is a man and is in this room, etc.’¹⁰

Being a man and being in this room are genuine concepts. They are not grammatical categories. Unlike mathematical ones, non-mathematical *Zahlangaben* contain an element of uncertainty [Unbestimmtheit]¹¹. For example, saying that ‘I know three men are in this room, but I do not know which’ makes sense.

8. PR §115

9. PR §116

10. PR §115

11. PR §115

Es fällt auf, daß der Satz von den 3 Kreisen nicht die Allgemeinheit oder Unbestimmtheit hat, die ein Satz von der Form $(x, y, z) \cdot fx \cdot fy \cdot fz$ besitzt. In diesem Fall kann man nämlich sagen: Ich weiß zwar daß 3 Dinge die Eigenschaft f haben, weiß aber nicht *welche*. Im Fall von den 3 Kreisen kann man das nicht sagen. [PR §115 p. 126]

It is plain that the proposition about the three circles isn't general or indefinite in the way a proposition of the form $(x, y, z) \cdot fx \cdot fy \cdot fz$ is. That is, in such a case, you may say: Certainly I know that three things have the property f , but I don't know *which*; and you can't say this in the case of three circles. [PR §115 p. 136]

Es hat also auf den Satz " $(x) \cdot fx$ " nicht in allen Fällen die Frage einen Sinn, "*welche* x befriedigen f " [PG Pt. I Appendix 2. p. 398]

So it does not always makes sense when presented with a proposition " $(x) \cdot fx$ " to ask "*Which* x s satisfy f ?" [PG Pt. I Appendix 2. p. 204]

In mathematics, knowing 'how many' implies knowing 'which'.¹²

In arithmetic, the question 'How many X s are such that Y ' only makes sense in the case where X is the specification concept 'unit' and Y is a calculation concept. In consequence, it makes sense to ask how many units are in $3 + 4$, but not how many numbers or additions are there in arithmetic.

Begriffswörter in der Mathematik: Primzahl, Kardinalzahl, etc. Es scheint darum inmittelbar Sinn zu haben, wenn gefragt wird: "Wieviel Primzahlen gibt es?" ("Es glaubt der Mensch, wenn er nur Worte hört, . . .") In Wirklichkeit ist diese Wortzusammenstellung einstweilen Unsinn; bis für sie eine besondere Syntax gegeben wurde. [PG PT. II §24 p. 738]

In mathematics there are concepts words: cardinal number, prime number, etc. That is why it seems to make sense straight off if we ask "how many prime numbers are there?" (Human beings believe, if only they hear words . . .) In reality this combination of words is so far nonsense; until it's given a special syntax. [PG Pt. II §24 p. 375]

It is very different to ascribe a cardinal number to a mathematical grammatical category than to a genuine concept. Counting objects which fall under genuine concepts, like apples on a table or shirts in a chest is radically different from counting mathe-

¹². PR §115

mathematical entities, like circles on a geometrical plane or prime numbers in a class. Counting the members of genuine concepts exclusively results in genuine numerical propositions, not mathematical ones. Accordingly, it is not a calculation. Counting mathematical entities is.

If cardinality were a concept's property, as Frege professed, then ascribing a cardinal number to a mathematical concept would be describing it. However, mathematical *Zahlangaben* are not descriptions of this sort.¹³ They are what Wittgenstein called *internal* descriptions.¹⁴ Cardinality adds nothing to a mathematical concept.¹⁵ The mathematical concept already includes its own cardinality. Consequently, the question *How many?* becomes "a straightforward problem."

Wenn ich jemanden frage, 'Wieviele Primzahlen gibt es zwischen 10 und 20?', so wird es sagen ich weiß es nicht im Augenblick, aber ich kann es jederzeit feststellen. Denn ist ja gleichsam schon irgendwo aufgeschrieben. [PR §114 p. 124]

If I ask someone, How many primes are there between 10 and 20?, he may reply, 'I don't know straight off, but I can work it out any time you like.' For it's as if there were somewhere where it was already written out. [PR §114 p. 134]

¹³. In strict sense, for Wittgenstein, mathematical *Zahlangaben* are not *external* descriptions, but *internal* ones. However, internal descriptions do not ascribe properties to objects. Hence, mathematical *Zahlangaben* –as internal descriptions– do not ascribe cardinality as a property to concepts as objects. Unless one wants to introduce the external/internal distinction to the realm of properties. Then, it would make sense to say that cardinality is an external property of genuine concepts and an internal property of grammatical and mathematical concepts. However, whatever one may win in with such talk is questionable.

¹⁴. On the notion of 'internal description' and Wittgenstein's use of the distinction between internal and external descriptions to make sense of the distinction between mathematical and genuine *Zahlangaben*, cf. Chapter 1, part II, section E: "Mathematical Objects and Concepts."

¹⁵. Wittgenstein went as far as to suggest that the question *how many* only made sense of genuine concepts, not mathematical (or grammatical) ones. In *Philosophical Remarks* §99, Wittgenstein writes, "For instance, does it mean anything to say 'a and b and c are three objects'? I think obviously not." It is hard to phantom Wittgenstein's point in this respect, except stressing the drastic differences between mathematical and genuine *Zahlangaben*.

B. Arithmetical Propositions

One finds three different characterizations of arithmetical statements in Wittgenstein's writings of the early thirties: as calculation statements, specification statements, and mathematical *Zahlangabe*. However, these are not three different kinds of statements, but three different approaches to arithmetical propositions. In Wittgenstein, there is a grammatical unity, not only to arithmetical, but to all mathematical statements. In the end, these three approaches collapse in one single grammatical kind.

Calculation statements connect calculations with their results. Equations and logical theorems are paradigmatic examples of calculation statements. Constitutive statements are statements of the form '*a* is a *B*' where *a* and *B* are mathematical categories. Mathematical *Zahlangaben* are statements of the form 'there are *n* *As*', where *n* is a cardinal number and *A* is a mathematical concept. These seem to be different sort of arithmetical propositions because each seems to be true in a different sense. Calculation statements are true if and only if they connect a calculation with its correct result. Constitutive statements of the form '*a* is a *B*' are true if and only if the category *a* is included in *B*, that is, if every expression in *a* is included in *B*. Finally, a mathematical *Zahlangabe* of the form 'there are *n* *As*', is true if the display of concept *A* presents *n* elements. However, despite their superficial differences, all mathematical statements share the same grammatical features.

Mathematical *Zahlangaben* are calculation statements, because counting mathematical entities, unlike counting genuine objects, is a calculation. A mathematical *Zahlangabe* of the form 'There are '*n*' *As*' is actually a calculation statement. It connects the calculation 'counting the *As*' with its result '*n*'. It says that counting the *As* results in '*n*'. Calculation statements are also constitutive statements. Since every calculation concept '*C*' is a grammatical category, the concept 'being the result of *C*' is one, too. In consequence, the calculation statement connecting calculation '*C*' with result '*a*' is equal to the constitutive statement '*a*

is B ' where ' B ' is the concept 'being the result of C '. Consequently, all mathematical statements are constitutive statements. They all express grammatical relations between categories. All mathematical statements are 'internal descriptions'.

Mathematics is calculation, because it involves nothing more than displaying the extension of mathematical concepts. Mathematics is grammar, because mathematical concepts are grammatical categories. Displaying their extension is constructing it in obedience of the calculus' grammatical rules. Checking if such a display produces a given expression type, i.e. if it obeys the rules that define the type, is all it takes to determine the truth of a mathematical propositions. For example, checking if an arithmetical equation of the form ' $C = a$ ' is true consists in (i) displaying the extension of the concept 'result of C ', and (ii) checking if the extension includes a . Displaying the extension of 'result of C ' is calculating C . Checking if constitutive statement '5 is a natural number' is true involves displaying the (extension of) number five and checking whether such construction follows the rules that define natural numbers.¹⁶ This process is grammatical, because the criteria for displaying the extension of a mathematical concept is grammatical. For example, the rules of addition constitute the criteria for displaying the extension of the calculation concept 'addition'. In every case, performing a mathematical calculation equals producing expression tokens of a determined type. In consequence, every constitutive statement is also a calculation statement, where displaying the relevant, mathematical concept is the calculation. A constitutive statement of the form ' a is a B ' is a calculation statement connecting a calculation - the

¹⁶. Wittgenstein's phrasing of this process is misleading, for displaying the extension of a concept is not listing all of its members. Calculating whether 5 is a number or not is not displaying the extension of a mathematical category - 'natural number' - to see if numeral 5 appears in that extension. It does not consist in, first, producing the infinite collection of natural numbers and, then, check them one-by-one to see if 5 is one of them. It

display of the extension of concept B – with its result. The apparent diversity behind mathematics dissolves. All mathematics is calculation. All calculation is grammatical.

IV. Conclusions

In 2.1512 of the *Tractatus*, Wittgenstein wrote that a proposition is laid against reality like a ruler. In the early thirties, Wittgenstein wrote that systems of propositions, not single propositions, work as rulers to measure reality. Instead of atomic and molecular propositions, sets of coordinates describe reality. Ascribing an attribute to an object fixes a coordinate of reality. Hence, it can only be determined *once*. Assigning a value to a coordinate of reality fully determines it.

Given a grammatical category G and a context C , the set $\{C(e) \mid e \in G\}$ is a system of propositions if even though only one of the propositions in the system is the case, they are all possible; i. e. they all make sense. In other words, every predication of a grammatical concept has one and only one value. The measuring scale for the value of these concepts is a grammatical convention. The system of cardinal numbers is a scale for calculating the value of the grammatical concept ‘quantity’. In consequence, it is a grammatical convention.

Count objects which fall under genuine concepts is radically different from counting mathematical entities. Mathematical *Zahlangaben* are calculation statements, because counting mathematical entities, unlike counting genuine objects, is a calculation. Ascribing cardinality to a mathematical concept does not describe any of its properties, as Frege professed, but displays its extension.

A mathematical *Zahlangabe* of the form ‘There are n As’ is a calculation statement, expressing that n is the result of counting the As. Calculation statements, in turn, are a special case of constitutive statements. Since a calculation concept C is a grammatical

category, the concept ‘result of C ’ is, too. In consequence, a calculation statement, connecting a calculation C with its result a , equals a constitutive statement of the form ‘ a is B ’ where B is the concept ‘being the result of C ’. In conclusion, all mathematical propositions are constitutive propositions expressing grammatical relations between categories

Mathematics consists entirely of displaying mathematical concepts’ extensions. Calculating is following the grammatical rules determining the extension of a mathematical concept. Checking the truth of a calculation proposition ‘ $C = a$ ’ is checking if the displayed extension of concept ‘result of C ’ includes an expression of type ‘ a ’. Displaying the extension of ‘result of C ’ is performing calculation ‘ C ’. Checking the truth of constitutive statement ‘ a is a C ’ is displaying an expression of type ‘ a ’ and checking whether such process obeys the rules that define the extension of ‘ C ’. This process is grammatical, because the criteria for displaying the extension of a mathematical concept is grammatical, too. In every case, performing a mathematical calculation is producing expression tokens of a determined grammatical type. In consequence, every constitutive statement is, in a certain sense, also a calculation statement where the calculation displays the relevant, mathematical concept’s extension. A constitutive statement of the form ‘ a is a B ’ is a calculation statement connecting a calculation – the display of the extension of concept B – with its result. In consequence, all mathematical propositions are calculations, and all calculation is grammatical.

Chapter 8

Grammatical Necessity

I. Introduction

Despite the ‘linguistic turn’ in philosophy at the beginning of the twentieth century, philosophers insist on stressing the boundaries between linguistics and their discipline, instead of taking advantage of their overlap. Philosophers of all sorts are reluctant to recognize the relevance of linguistic studies for their field. Even philosophers of language like Gilbert Ryle¹ and Stanley Cavell² have claimed that the results of linguistic science offer nothing to philosophy. This view results in the common assumption that when a philosopher like Wittgenstein talks about ‘grammar’ or ‘syntax’, he cannot refer to the linguistic disciplines of the same name. He must refer to some esoteric *logical* syntax or *deep* grammar. This dissertation aims at dispelling this common misconception. When referring to mathematical propositions as grammatical, Wittgenstein does not use the term ‘grammatical’ in a radically different way than linguists.

According to the maxim, *valet illatio ab esse ad posse*: The best way to show that something is possible is by doing it. In this case, showing how grammatical analysis can yield mathematical results is the best way to demonstrate that mathematics may be grammar. Chapter 6 pursued this goal. It developed familiar theorems of arithmetic out of a grammatical analysis of the use of numerical expressions, both in calculation and in natural language. Its results showed formally that numerical calculi not only constitute grammatical systems, but that they belong to the grammar of ordinary language. It proved that the

1. Gilbert Ryle, “Ordinary Language,” *Philosophical Review* LXII (1953) : 167-186. “Use, Usage and Meaning,” *Proceedings of the Aristotelian Society* supp. vol. XXV (1961) : 223-230.

2. Stanley Cavell, “Must we mean what we say?” *Inquiry* 1 (1958) : 172-212. “The availability of Wittgenstein’s later Philosophy” *Philosophical Review* LXXI (1962) : 67-93.

grammar of any language with numerical expressions must include rules with a natural mathematical interpretation. Yet, to make the case that grammatical analysis, in fact, supports mathematical conclusions, responding to the various criticisms in the secondary literature is also necessary.

The present chapter defends Wittgenstein's position against criticisms. The first part focuses on some general arguments against the grammatical nature of mathematical propositions. These arguments support the claim that grammatical propositions, unlike mathematical ones, describe the usage of words. These arguments conclude that grammatical propositions are contingent, while mathematical ones are necessary. The first part of this chapter presents a defense of Wittgenstein's grammatical account of mathematics against the aforementioned objections. This defense is based on Morris Lazerowitz's "Necessity and Language", Zeno Vendler's "Linguistics and the *a-priori*" W. E. Kennick's "Philosophy as Grammar"³, and J. Michael Young's "Kant on the Construction of Arithmetical Concepts."⁴ The argument shows that the objections raised against Wittgenstein equivocate on the meaning of the adjective 'grammatical'. The second part of the chapter deals in closer detail with the Quine/Carnap debate and its relevance to Wittgenstein's grammatical account of mathematics during the early thirties.

3. W. E. Kennick, "Philosophy as Grammar" in G. E. Anscombe & M. Lazerowitz: *Ludwig Wittgenstein, Philosophy and Language* (Bristol: Thoemmes Press, 1996)

4. J. Michael Young, "Kant on the Construction of Mathematical Concepts" *Kant-Studien* 73 (1982): 17-46. Michael Young's article differs from those of Vendler, Lazerowitz and Kennick's in that it focuses on questions in the philosophy of mathematics. Michael Young uses an argument similar to the present one to "show that Kant is right in thinking that to ground *a priori* judgements, at least in arithmetic, upon ostensive constructions" is possible [p. 17] Primarily, Kant's and Wittgenstein's position regarding the construction of arithmetical concepts differ, because the rules of calculation that Kant refers to as 'the universal conditions of construction', are distinct from the concepts whose constructions they govern. Cf. *Ibid.* 28, 29.

II. On the Grammatical Nature of Mathematics

To understand the grammatical nature of mathematical propositions, it would be helpful to translate the formal results from the previous chapters into a more informal comparison between mathematical and more obvious grammatical propositions. This comparison will take place in two parts. The first part will develop the different senses in which statements are said to be ‘grammatical’. The second part will make an analogy between these grammatical statements and properly mathematical ones. This analogy’s goal is to clarify the sense in which mathematical propositions are grammatical. It also sets the basis to discuss the four general arguments against the grammatical nature of mathematics.

A. Arguments against the Grammatical Nature of Mathematics

In general, four major arguments are raised against the claim that mathematical propositions are grammatical:

1. Linguistic practice is an empirical fact. Hence, grammatical propositions about verbal usage are empirical generalizations and, consequently, not necessary. In contrast, mathematical propositions are necessary.
2. Understanding grammatical propositions as those that describe the usage of words implies that grammatical propositions are not necessary. Negating a true proposition about verbal usage is not a contradiction, but a false proposition.⁵ Morris Lazerowitz, presents this objection as follows:

The negation of a true verbal proposition is a false verbal proposition, but not a proposition which could not, in principle, be true. . . . To use an expression of Wittgenstein’s, we know what it would be like for a verbal *proposition*, which happens to be true, to be false. By contrast we do not know what it would be like for a false arithmetical proposition to be true, for example, for $4 + 3$ to be less than 7.⁶

⁵ Morris Lazerowitz, introduction to M. Lazerowitz and Alice Ambrose eds. *Essays in the Unknown Wittgenstein* (Buffalo: Prometheus Books, 1984), 6.

⁶ Morris Lazerowitz: “Necessity and Language“ (Lazerowitz 1984, 235)

3. If grammatical propositions record the usage of words, they must describe particular words in a particular language. Mathematical propositions do not, in general, say anything about vocabulary. Furthermore, if grammatical statements are about words, morphemes, etc. and their uses, then their truth depends on the existence of these linguistic entities. If, as Wittgenstein contended, ' $3 + 4 = 7$ ' does not deal with abstract entities called numbers 3, 4 and 7, but with numeral types '3', '4' and '7', then it features a commitment to the existence of these numerals.
4. Finally, mathematical truths do not depend on the language expressing them. Hence, mathematical propositions cannot be grammatical. The peculiarities of one language are not sufficient to solve genuine mathematical problems.

The objection that grammatical propositions, unlike mathematical ones, are language-specific is at least as old as Moore's notes on Wittgenstein's lectures of Lent and May terms of 1930. He reports that Wittgenstein stated,

. . . the proposition 'red is a primary color' was a proposition about the word 'red'⁷

Immediately after, Moore observed that,

. . . if he had seriously held this, he might have held similarly that the proposition or rule ' $3 + 3 = 6$ ' was merely a proposition or rule about the particular expressions ' $3 + 3$ ' and '6'.⁸

Moore himself recognized the absurdities that his interpretation of Wittgenstein implied, when he commented,

. . . he cannot have held seriously either of these views, because the *same* proposition which is expressed by the words 'red is a primary color' can be expressed in French or German by words which say nothing about the English word 'red'; and similarly the *same* proposition or rule which is expressed by ' $3 + 3 = 6$ ' was undoubtedly expressed in Attic Greek and in Latin by words which say nothing about the numerals '3' and '6'. And

7. *Philosophical Papers*, 275 quoted in (Lazerowitz 1984, 16-17)

8. *Ibid.*

this was in fact what he seemed to be admitting in the passage at the end of (I).⁹

Mathematical propositions do not say anything about vocabulary. Furthermore, their truth does not depend on the language expressing them. For Moore, this meant that they cannot be grammatical.

Grammatical statements express language rules, even if they do not mention any explicitly linguistic entities like morphemes, words, etc. Still, mathematical propositions are categorical in their necessity. The following sections deal with this apparent tension.

B. Grammatical Statements

Consider an obviously grammatical transition of ordinary English language: the transition from passive to active forms. This transition may easily be formulated as a syntactic rule:

- (1) The passive form of an active sentence $a B c d$ (where a is the sentence's subject, B its verb, c the verb's direct compliment, and d is the string of indirect compliments of B) is the string $c BE(c/B) PP(B) 'by' a d$, where $BE(c/B)$ is the conjugation of the verb 'to be' in the number of c and the time of B , and $PP(B)$ is the past participle form of verb B .

This rule allows us to transform active sentence (2) into passive sentence (3):

- (2) Many persons have attended the dance marathon since its inception.
 (3) The dance marathon has been attended by many persons since its inception.

⁹ *Philosophical Papers*, 41 quoted in (Lazerowitz 1984, 17). Leaving aside for a moment the possibility that Wittgenstein might have actually said that 'red is a primary color' is *about* the word 'red', Moore's assumption that, if a grammatical proposition is about words, it must be about the words that occur in it is surprising. In many cases, grammatical propositions address the correct use of terms not in them. 'Number words can function as adjectives' is a grammatical proposition about the correct use of number words. Still, no number words occur in it. On the other hand, "'Spanish' is spelled with capital 'S'" is about the spelling of the word 'Spanish' in it. In general, external, grammatical statements are about the use of words in them, while internal ones are not.

This transformation may also be expressed in a single sentence (just like the conditionalization of a *modus ponens*):

(4) If many persons have attended the dance marathon since its inception, then the dance marathon has been attended by many persons since its inception.

This illustrates the double nature of grammatical application. For Wittgenstein, grammatical rules may be applied both in the formation (as in 4) and transformation (as from 2 to 3) of acceptable strings.¹⁰

Finally, the following statement also expresses this application of the rule expressed in (1):

(5) The passive form of “Many persons have attended the dance marathon since its inception” is “The dance marathon has been attended by many persons since its inception.”

Since the rule for the transformation from active to passive is grammatical, statements (1), (4) and (5), and the transition between statements (2) and (3) may correctly be called grammatical. However, they are grammatical in a different sense. Their relation to the grammatical rules is different. Both (4) and the transition from (2) to (3) are *Anwendung* of the grammatical rule. Statements (1) and (5), on the contrary, are expressions of the rule. As such, they are *about* the grammatical rule. In consequence, they are *heteronomous* grammatical statements. Statement (4), in contrast, is a necessary *autonomous* grammatical statement. When Wittgenstein talks about grammatical statements, he is mostly referring to statements like (4), that is autonomous grammatical statements which do not express or are about any grammatical rule, but display it in its application.

This difference becomes essential once questions of truth and necessity come into play. It is clear that the question of truth can only be brought about statements and not about

¹⁰. Hence, he does not make a distinction between formation and transformation rules, as most traditional grammarians do.

transitions like that between (2) and (3). In those cases, the question of grammatical necessity is not that of necessary truth, but necessary transition. This difference will become essential when dealing with criticisms of grammatical necessity, like that of Quine. At the moment, this section will center on statements like (1), (4) and (5).

The most obvious criticisms to the necessary nature of grammatical statements focus on statements like (1), also called *external*¹¹ or *explicitly*¹² grammatical statements. These statements, as descriptions of grammatical rules, are language-dependent and contain ontological commitments that render them not necessary. Wittgenstein has no problem with these criticisms in so far as he also considers statements like (1) to be not necessary. For Wittgenstein, a statement like (1) is not grammatical, but *about* grammar. It is not completely clear from Wittgenstein's writings whether statements like (5) are also grammatical. However, the issue is minor. There is a unique grammatical rule expressed in (1) and (5) and displayed in (4) and the transition from (2) to (3). This rule is the real grammatical proposition.

11. This nomenclature originates in the work of Zeno Vendler's *Linguistics in Philosophy* (Ithaca: Cornell University Press, 1967) 147-171. According to Vendler, a grammatical statement is *external* if it mentions a word, morpheme or any other linguistic entity, and says something about its use. Otherwise, it is *internal*. Consider some examples. Consider the statement "'Spanish' is spelled with a capital 'S'." This grammatical statement is external, because it mentions the word 'Spanish' and it says something about its use: that it is spelled with a capital 'S'. Now look at one example of internal, grammatical statements: 'Names of languages are capitalized'. This statement does not mention any words, but still expresses a grammatical rule in grammatical vocabulary. Other examples of external, grammatical statements are: "'Dog' is a noun" and "the gerund of 'walk' is 'walking'". Examples of internal, grammatical statements are: 'Number words function as adjectives' and 'Noun and adjective must agree in gender'. In Chomskian grammars, this distinction corresponds to terminal (external) rules, and non-terminal (internal) rules (of transformation and formation). In the formal reconstruction presented in chapter 4, grammatical propositions that express relations within expressions or between expressions and categories are *internal*, while propositions that express relationships among categories are *external*.

12. This convention is present in the work of W. E. Kennick, "Philosophy as Grammar" in G. E. Anscombe & M. Lazerowitz: *Ludwig Wittgenstein, Philosophy and Language* (Bristol: Thoemmes Press, 1996) and Morris Lazerowitz, "Necessity and Language" in M. Lazerowitz and Alice Ambrose eds. *Essays in the unknown Wittgenstein* (Buffalo: Prometheus Books, 1984). However, it does not completely match with Vendler's notion of 'external grammatical' statement. For Kennick and Lazerowitz, a statement is explicitly grammatical if its vocabulary is grammatical, and implicitly grammatical if it does not include grammatical terms, "but still expresses a rule, convention, or decision about verbal usage." [Lazerowitz. Ibid 142].

Consider, now a mathematical transition, for example, the addition of two numerals under 100. The rule that governs such calculation can be expressed the following way:

(6) The addition of two numerals a b and c d is the string $R(((R(b + d) + a) + c) ((R(b + d) + a) + c) (b + d)$, where $R(n) = '1'$ if $n < 10$ and $R(n)$ is the empty string otherwise.

This rule applies to the addition of 27 to 34. This calculation may be represented in the following way:

(7)

$$\begin{array}{r} 1 \\ 27 \\ +34 \\ \hline 61 \end{array}$$

This calculation is also expressed in the form of an equation as:

(8) $27 + 34 = 61$

Mathematics also features the double nature of grammatical application presented above for the case of natural language grammar. For Wittgenstein, grammatical rules may be applied both in the formation (as in 8) and transformation (as in 7) of expressions.¹³

The application of this rule to numerals '27' and '34' may also be expressed in the following statement:

(9) The result of adding 27 to 34 is 71.

Since the rule governing the calculation is grammatical statements (6), (8) (9), and display (7) are also grammatical. However, just like in the case of (1) to (5), they are all grammatical in different senses. Their relation to the mathematical rule is different.

(7), (8) and (9) all express the same calculation. Yet, when people think about mathematical statements, expressions like (8) most typically come to mind. For Wittgenstein, however, the calculation is displayed in (7) as well as in (8). A mayor difference is that

¹³. The transformed expressions need not be full statements in the case of internal *Anwendung*.

strings like (8) have the further disadvantage of looking too similar to natural language statements. It is customary to call (8) a mathematical *statement*. However, it is important not to think that, as a statement, it must be about something. Furthermore, it is also important not to infer that its about the addition, either as a calculation or as a mathematical operation. The relation between calculation and mathematical statement is not one of aboutness, but of trace. Statement (8), just like display (7), is the trace left by the calculation. Statement (9), in contrast, expresses this same calculation externally. Unlike (7) and (8), (9) is not a trace of the calculation. It expresses a genuine proposition. This proposition is not the calculation itself. It is *about* the calculation. Even if it does not include explicit mention of numerals, it still lacks the autonomy of belonging to the calculus as (7) and (8) do. In that sense, it is similar to (6). Both (6) and (9) are external mathematical statements. Their meaning is not an autonomous mathematical proposition, but a description of it.

The common criticisms to the grammatical nature of mathematical propositions are dissolved by paying closer attention to the analogy between statements (1) to (5) and (6) to (9). In mathematics, as in ordinary natural language grammar, it is very important to distinguish between grammatical statements and statements *about* grammar. Mathematical statements like (8) and (7) are grammatical, yet they are not about grammar. In strict sense, they are not about anything.

From (7) to (9), there is a unique calculation and, in consequence, a unique mathematical proposition. It is displayed in (7) and (8), but described in (9). Questions about the truth or necessity of mathematical propositions commonly stem from misguided analogies between mathematical statements and descriptive ones. These analogies conceal important differences between the descriptive and mathematical propositions behind the statements. Most of all, they hide the important difference between displaying a rule by following it and describing it.

C. Grammatical Propositions as Rules

As the second appendix to the *Philosophical Remarks*, the editors printed Friedrich Waisman's record of a conversation at Schlick's house, on December 30, 1930, where Wittgenstein drew an analogy between the necessity of a chess-proposition like 'I can force mate in 8' and that of an arithmetical equation. The analogy is based on the simple fact that, besides languages and calculi, other rule-governed practices have produced a specific vocabulary for expressing their rules. In chess, for example, expressions like 'pawn', 'opposite piece', etc. are part of chess vocabulary. Knowing the rules of chess does not only involve learning the permissible moves for each piece, the winning positions, etc. It also requires learning the names of the pieces: what is a check-mate and the like. In other words, knowing chess involves learning the *vocabulary* of the game. This vocabulary is given by the game's 'constitutive rules'.¹⁴ In strict sense, these rules do not say anything about how to play the game, but assist on the understanding of the other rules. They define the meaning of terms within the game.¹⁵

Consider now, a statement about chess in chess vocabulary. For example: "No two pawns of the same color can be in the same column without having captured an opposite piece." This statement expresses a necessary truth, precisely because it uses chess vocabulary ('pawn', 'capture', etc.). Any acceptable interpretation of this statement must comply with its terms' meanings. The constitutive rules of chess determine these meanings. Hence, any possible interpretation of the statement must already accept those rules. In the conceptual framework that the constitutive rules create, the statement in question expresses a

14. Do not mistake the constitutive rules of chess through which one learns its vocabulary with those ostensive statements assigning pieces' roles to different material objects. An ostensive statement indicating which object, for example, will be a pawn and which one a bishop is not a rule of the game of chess.

15. Describing the game of chess without using these words or some equivalent may be possible, but it would be extremely complicated and artificial.

necessary truth. Since the statement is true according to those rules, its truth is necessary. ‘No two pawns of the same color can be in the same column without having captured an opposite piece’ is necessary.

Someone may say that this proposition is not necessary, insofar as its truth depends on the rules of the game existing as they do, and this is not a necessary fact. The rules of chess could be otherwise, indeed. However, for two pawns of the same color to be in the same column without one capturing an opposite piece is still impossible. For this to happen, the constitutive rules for what is a pawn or what is to capture an opposite piece would have to be different. ‘Pawn’ and ‘to capture an opposite piece’ would have to mean something else. But, in that case, the statement ‘No two pawns of the same color can be in the same column without having captured an opposite piece’ would also mean something different. It would now express a false proposition. However, this false proposition would not be the original proposition. Whatever “no two pawns of the same color can be in the same column without having captured an opposite piece” would mean in this bizarre interpretation of its terms would be false. Yet, it would not be false that no two pawns of the same color can be in the same column without having captured an opposite piece.

If different games had the same vocabulary but different rules, it would be necessary to add a clause to every internal statement indicating in what game to interpret the sentence. If a game different from chess used chess terms but had different rules, it would be necessary to add the clause ‘in the game of chess’ to every chess statement. However, in principle, natural language excludes this possibility. The language of the grammatical statement already suggest the system of rules for interpretation. In Vendler’s words:

“in saying “One cannot know something false,” I am talking English, so the possibility of interpreting the statement according to the rules of some other language does not arise. To say things like “having a mistress was respectable in Old English but not in current English” is to make a bad joke.

. . . To conclude, a statement such as “One cannot know something false” is not true in English or for English; it is absolutely and categorically true.”¹⁶

The truth of grammatical statements depends on language the same way every other statement does. Every statement is true or false, according to the interpretation rules of the language. Grammatical statements are no different. The only difference is that they determine themselves the rules of their interpretation. This makes them true not only in the language of which they are rules, but in any language “provided they are well translated.”¹⁷

D. Grammar and Vocabulary

The necessity of a chess rule like ‘No two pawns of the same color can be in the same column without having captured an opposite piece’ does not require that those particular spatial and temporal objects called ‘pawns’ exist. In this sense, it does not require pawns existing. However, in another sense, it requires the existence of pawns, indeed. If pawns did not exist in chess, that is, if the game of chess were played without pawns, the proposition would be nonsense. In this sense, the proposition requires the existence of pawns to be meaningful. There is no contradiction here, just an equivocation in the understanding of pawns and their existence. In chess, ‘pawns’ refers both to a kind of piece in the game, and to those material objects playing their role in particular chess matches. The rules of chess determine the pawns’ essence in the first sense. However, they are indifferent to pawns in the second sense. In consequence, they require the existence of pawns as pieces defined in the game, but not as material objects. A chess rule does not refer to pawns as spatio-temporal entities existing outside the game. The rule refers to pawns as *pieces* in the game.

¹⁶. (Vendler 1967, 24)

¹⁷. Ibid. 26

The rules of the game completely define its pieces. ‘Being a pawn’ is playing a certain role in the game a chess.¹⁸

Es ist übrigens sehr wichtig, daß ich den Holzklötzchen auch nicht ansehen kann, ob sie Bauer, Läufer, Turm etc. sind. Ich kann nicht sagen: das ist ein Bauer *und* für diese Figur gelten die und die Spielregeln. Sondern die Spielregeln *bestimmen* erst diese Figur: der Bauer *ist* die Summe der Regeln, nach welchen er bewegt wird (auch das Feld ist eine Figur), so wie in der Sprache die Regeln der Sprache das Logische im Wort bestimmen.[PR Appendix II, p. 315]

Besides, it is highly important that I can’t tell from looking at the pieces of wood whether they are pawns, bishops, rooks, etc. I can’t say: that is a pawn *and* such and such rules hold for this piece. No, it is the rules alone which *define* this piece: a pawn *is* the sum of the rules for its moves (a square is a piece too), just as in the case of language the rules define the logic of a word. [PR Appendix II, p. 328]

Similarly, the necessity of a grammatical statement like ‘adjectives cannot modify verbs’ is not contingent on the material existence of adjectives and verbs, because it does not refer to them as spatio-temporal objects, ink marks on paper, but as grammatical categories. Wittgenstein calls grammatical statements ‘internal descriptions’, because they address grammatical categories. They are not about objects. The aforementioned statement is not about any objects called ‘adjectives’, but about the grammatical category ‘adjective’. It states its relationship with the grammatical category ‘noun’.

By analogy, saying that a mathematical proposition like ‘there is no integer between three and four’ relies on the integers three and four existing is also equivocal. Since a mathematical statement like ‘ $3 + 4 = 7$ ’ says that the correct result of adding three to four is seven, its truth does not depend on any spatio-temporal objects or events. As an internal, grammatical statement, it does not refer to additions as spatio-temporal events, but as calculations defined within the mathematical system. It is not contingent on particular instances of numerals ‘3’ and ‘4’ existing either. It does not refer to them. It refers to the

¹⁸. The rules determine its essence. In consequence, whatever they say is essential.

numbers three and four. These numbers are the numerals' grammatical categories in the calculus.

Saying that the proposition 'No two pawns of the same color can be in the same column without having captured an opposite piece' is about the word 'pawn' in English is as absurd as saying that an arithmetical equation like ' $3 + 4 = 7$ ' is about the English word 'seven'. The first proposition is about the pawn piece the sum of chess rules define. Similarly, the arithmetical equation is about the number seven the arithmetical rules define. Michael Young writes,

In calculating we do deal with a particular collection of characters or marks, say those that we have written on a piece of paper, but it should be clear that we do not deal with them as perceptual objects in their own right, attributing to them whatever properties they might happen to exhibit. If one '6' happens to be larger than another instance, or to have a different color or shape, we recognize that this is quite irrelevant. We treat the characters that we intuit merely as instances of the Arabic numerals, ignoring everything else about them.¹⁹

The way a genuine proposition like 'the current king of France is bald' relies on the existence of the current king of France (or a chess statement requires the existence of chess pieces) is significantly different from the way an arithmetic proposition relies on numbers existing. Without a current French king, any external description of the current king of France would be nonsensical. Without chess, the statement 'No two pawns of the same color can be in the same column without having captured an opposite piece' would not be false, but absurd. Without the existence of numbers 3, 4 and 7, ' $3 + 4 = 7$ ' would be just a senseless string of marks on paper. However, the similarities stop here. Unlike genuine propositions or chess rules, mathematical propositions are autonomous. Since the calculus is its own internal *Anwendung*, mathematical statements do not describe mathematical rules, they are themselves mathematical rules.

¹⁹. (Young 1982, 25)

Mathematical propositions are syntactically necessary, because they cannot belong to a calculus and be false. They require no more than their own calculus. Without numbers, arithmetical equations would not exist. If 7 were not a number, $3 + 4 = 7$ would not be an arithmetical equation. Nevertheless, if the arithmetical equation exists, the numbers and operations involved in it exist too. The arithmetical equation $3 + 4 = 7$ is grammatically necessary, because its existence in the calculus guarantees the existence of its terms. The existence of the equation $3 + 4 = 7$ in the calculus guarantees that 3, 4 and 7 also exist.

The equation $3 + 4 = 7$ guarantees more than 3, 4 and 7 existing in the calculus. It also guarantees that adding three to four is seven. Otherwise, the equation would not belong to the calculus, either. It would not be an arithmetic proposition. If adding three to four was not seven, ' $3 + 4 = 7$ ' would not be a false arithmetical proposition. It would not be an arithmetic proposition at all. The connection between a genuine proposition and whatever it is about differs radically from the connection between a mathematical proposition and a calculation. Genuine propositions describe possible states of affairs. Mathematical propositions do not describe calculations. They are themselves calculations. The truth of a genuine proposition like 'the cat is on the mat' requires the cat being on the mat. A mathematical proposition does not require anything that it does not construct for itself.

III. Wittgenstein's Syntactic Necessity as Analyticity

A. Brief Historical Background

According to the middle Wittgenstein, internal descriptions ascribe essential properties to objects, while external descriptions ascribe accidental properties.²⁰ A description is internal if the concept in the subject includes or implies the concept in the predicate. This

²⁰. PR §94.

characterization of internal descriptions is very close to that of analytical statements. Since Wittgenstein also includes mathematical statements among internal descriptions, this commits him to believe that mathematical statements are analytic.

Before the linguistic turn in philosophy at the end of the nineteenth century, Locke had already distinguished two kinds of analytic propositions. In *An Essay concerning Human Understanding* [pp. 306, 308], he distinguished between ‘trifling’ and ‘predicative’ propositions. Trifling propositions have the form ‘ $a = a$ ’, in which “we affirm the said term of itself.” In predicative propositions, “a part of the complex idea is predicated of the name of the whole.” For Locke, mathematical propositions are not analytic in either of these senses. After Locke, Kant added a new account of analyticity to Locke’s notion of trifling proposition. For Kant, an analytic judgement is (i) one whose subject concept contains its predicate concept, or (ii) one whose negation is a logical contradiction. By offering these two different accounts, Kant laid the foundations for what became the two main doctrines of analyticity in modern western philosophy.²¹ For Kant, a judgement is analytic if the subject’s concept contains the predicate’s concept. However, he allows for two possible interpretations of this ‘containment’, what Jerrold J. Katz in *The New Intentionalism* calls ‘logical-containment’ and ‘concept-containment’.²² Kant’s notion of ‘analytic’ fused these two notions, as they remained until Frege separated them. For Frege, Kant’s account of analyticity in terms of conceptual containment was a psychologistic error. In *The Foundations of Arithmetic* §3, Frege defines analyticity as ‘being a consequence of logical laws plus definitions without scientific assumptions’. Wittgenstein’s account of logical necessity in the *Tractatus* follows Frege away from the conceptual path and into logicism. This path leads from Wittgenstein directly into the Quine/Carnap controversy. At the end of

²¹. They could be called the ‘logician’ and the ‘idealist’ doctrines.

²². Jerrold J. Katz, *Mind*, New Series, vol. 101, issue 404 (Oct., 1992), 691.

his 1944 article on ‘Russell’s mathematical logic’ Kurt Gödel distinguishes two senses of ‘analyticity’.

As to this problem [if (and in which sense) mathematical axioms can be considered analytic], it is to be remarked that analyticity may be understood in two senses. First, it may have the purely formal sense that the terms occurring can be defined (with or without explicitly or by rules for eliminating them from sentences containing them) in such a way that the axioms and theorems become special cases of the law of identity and disprovable propositions become negations of this law. . .

In a second sense a proposition is called analytic if it holds “owing to the meaning of the concept occurring in it”, where this meaning may perhaps be undefinable (i.e., irreducible to anything more fundamental). [Note 47. The two significations of the term ‘analytic’ might perhaps be distinguished as tautological and analytic.]²³ According to Carnap, Wittgenstein’s *Tractatus* endorsed the view of mathematical propositions as analytic in the tautologous sense. However, by the beginning of the thirties, Wittgenstein’s view on the analyticity of mathematics had evolved from a purely formal notion into Gödel’s second sense.

After the *Tractatus*, considerations about color and the nature of space changed Wittgenstein’s mind about the logicist’s path. In the *Tractatus*, he had maintained that “there is only *logical necessity*” [6.375]. However, by the late twenties, he could hardly see how the *Tractatus*’ logical necessity could account for the necessity of such propositions as ‘The blue spot is not red at the same time’. In the early thirties, the notion of grammatical necessity had become a substitute for that of logical necessity in the *Tractatus*.

B. Carnap

The debate between Carnap and Quine – and, by extension, Tarski, Gödel, Dummett, Putnam, et. al. – concentrates on mathematics as part of the formal syntax of language. Because Carnap asserted that his thesis of mathematics as syntax sprang from Wittgenstein, taking a stance regarding this debate is critical. Clarifying whether or not he held a view like the one Carnap championed is vital, as is defending Wittgenstein against Quine’s criticisms.

Despite their mutual personal dislike,²⁴ Carnap always recognized Wittgenstein’s

²³. (Gödel: 1986, 139)

²⁴. On March 27, 1998, as part of an electronic exchange in the *Foundations of Mathematics* mailing list,

influence on this and other philosophical matters. In his “Intellectual Autobiography,” Carnap states that “Wittgenstein was perhaps the philosopher who, besides Russell and Frege, had the greatest influence on my thinking.”²⁵ From Carnap’s own appraisal, the sources of this influence were triple: (i) careful and intense reading of the *Tractatus* by the Vienna Circle, (ii) personal contact between Carnap and Wittgenstein from the Summer of 1927 to the beginning of 1929, and (iii) “Waismann’s systematic expositions of certain conceptions of Wittgenstein’s on basis of his talks with him.”²⁶

According to Carnap,

“The most important insight I gained from his [Wittgenstein’s] work was the conception that the truth of logical statements is based only on their logical structure and the meaning of terms. Logical statements are true under all conceivable circumstances; thus their truth is independent of the contingent facts of the world. On the other hand, it follows that these statements do not say anything about the world and thus have no factual content.”²⁷

From Wittgenstein, Carnap received the idea that logical truths are tautologies. In the *Tractatus*, Wittgenstein unsuccessfully argued for the tautologous nature of logical truth for the first time in the history of logicism.

However, the issue of logical truth is the source of both the main agreement and most important divergence between Carnap and Wittgenstein. According to Michael Friedman,

Neil Tennant wrote: “It is worth recording here that Carnap---perhaps the most influential member of the V.C. as far as philosophy of mathematics and foundations are concerned---and Wittgenstein did not hold one another in very high regard, despite Carnap’s always polite and forbearing published comments on the influence of Wittgenstein.” Tennant offers evidence documented from his study of the Carnap archives. Unlike the apparently personal attacks, Tennant documents the priority debate between Wittgenstein and Carnap in his paper “The Life and Work of the Early Carnap,” in N. Rescher ed., *Scientific Inquiry in Philosophical Perspective*, University Press of America, 1987, pp.261-280. To: parsons2@fas.harvard.edu Subject: “Re: FOM: Wittgenstein: two clarifications.” From: Neil Tennant <neilt@mercutio.cohums.ohio-state.edu>. Date: Fri, 27 Mar 1998 13:40:26 -0500 (EST). Sender: owner-fom@math.psu.edu.

25. (Schilpp 1963, 46)

26. Ibid. 28

27. Ibid. 25

This conception of the tautologous character of logical and mathematical truth represents Carnap, the most important point of agreement between his philosophy and that of the *Tractatus*. But there is also an equally important point of fundamental disagreement. Whereas the *Tractatus* associates its distinctive conception of logical truth with a radical division between what can be said and what can only be shown but not said – a division according to which logic itself is not properly an object of theoretical science at all – Carnap associates his conception of logical truth with the idea that logical analysis, what he calls “logical syntax,” is a theoretical science in the strictest possible sense.²⁸

In terms of the middle Wittgenstein, the main point of divergence between Carnap and Wittgenstein was the autonomous character of mathematics and grammar. Under the influence of Frege and Russell, Carnap was always convinced of “the philosophical relevance of constructed language systems.”²⁹ During his years in the Vienna Circle, Otto Neurath nurtured Carnap’s idea that a descriptive science of the structure of language – what would become the “Logical Syntax of Language” – was possible. Finally, Carnap’s study of Hilbert and his continuous talks with Tarski and Gödel convinced him of the philosophical power of meta-mathematics. By the time he had developed his theory of logical syntax, virtually all connection with Wittgenstein’s notion of tautology and analyticity seemed lost.³⁰ Most strikingly, Carnap’s logical syntax of language, unlike Wittgenstein’s grammar, had lost its autonomy.

Carnap conceived of philosophy as a descriptive, scientific enterprise geared towards formulating the logic of science in a precise meta-language.³¹ Instead of an indescribable, but displayable grammar, Carnap expresses his logical syntax in its own object language. Carnap uses Gödel’s arithmetization method to embed the syntactic meta-language in the

28. Michael Friedman, “Carnap and Wittgenstein’s *Tractatus*” (Tait 1997, 20)

29. (Schilpp 1963, 28)

30. (Friedman 1997, 23)

31. Rudolf Carnap, *The Logical Syntax of Language* (New York: Harcourt, 1937) §73.

object language (provided that the object language includes elementary arithmetic), allowing it to express its own syntax. However, it immediately follows from Gödel's work that, for a language containing classical arithmetic, 'truth' is a non-arithmetical predicate and thus, undefinable in the language itself. Carnap understood this and, hence, qualified his remarks on this method in *Logical Syntax*. Commenting on the Wittgenstein / Carnap connection, Michael Friedman interprets this as a point in favor of Wittgenstein's autonomous grammar over Carnap's logical syntax.³²

The failure of Carnap's attempt to syntactically define analyticity is a point in favor of the autonomy of mathematics. Carnap followed Wittgenstein's search for mathematics in the syntax of language. Natural language grammar contains embedded mathematical calculi. However, Carnap was wrong in thinking that mathematics describes this external *Anwendung* in a meta-language. Mathematics is autonomous. Every calculus is its own internal *Anwendung*. This internal *Anwendung* does not require a metamathematical formulation. The calculus is sufficient.

It is possible to describe a calculus' external *Anwendung* in a meta-language. Chapter 3 is an example of this. However, this description is *not* the calculus itself. Describing a syntax is substantially different from calculating. Unlike calculation, description is not autonomous. The truth of a descriptive proposition point outside the description itself. Calculation determines the correctness of its own propositions. The mere

³². "Carnap, characteristically, has transformed an originally philosophical point into a purely technical question. – in this case, the technical question of what formal theories can or cannot be embedded in a given object language. Considered purely as a technical question, however, the situation turns out to be far more complicated than it initially appears. . . For it turns out, again as a consequence of Gödel's researches, that it is as a matter of fact not possible in most cases of interest to express the logical syntax of a language in Carnap's sense in the language itself. . . Thus, the logical syntax in Carnap's sense for a language for classical mathematics can only be expressed in a distinct and essentially richer metalanguage; the logical syntax for this metalanguage can itself only be expressed in a distinct and essentially richer meta-metalanguage; and so on. . . Does this same situation does not represent the kernel of truth – from Carnap's point of view, of course – in Wittgenstein's doctrine of the inexpressibility of logical syntax?" (Friedman 1997, 35-36)

description of a calculus' external *Anwendung* cannot fully determine the correctness or incorrectness of its propositions. Gödel showed that Carnap's attempt failed technically. Wittgenstein showed that the project was also philosophically inadequate.

C. Quine

1. Two Dogmas and the Analytic Nature of Grammar

The linguistic doctrine of logical truth is sometimes expressed by saying that logical truths are true by linguistic convention.

Quine 1963, 391

The analytic/synthetic distinction has a long history in modern philosophy. According to Quine's "Two Dogmas of Empiricism", the writings of Leibniz, Hume and Kant foreshadow the contemporary distinction. However, both Hume's "relations of ideas" and Leibniz's "truths of reason" are quasi-psychological notions. It was Kant who first inserted language at the core of the philosophical characterization of analyticity. The idea of 'truths independent of fact' precedes Kant. Nonetheless, starting with him, these truths became also 'true by virtue of meaning'. The current notion of 'analyticity' originates in Kant. After the seminal work of Frege, analyticity secured a central place in contemporary philosophy of logic and mathematics. The discussion of analyticity in this century has grown largely from his conception. Nevertheless, Quine offered the principal arguments against the analytic/synthetic distinction, not in response to Frege, but in response to Carnap's *The Logical Syntax of Language*." Those arguments are so convincing that even today a large number of philosophers and mathematicians consider some of the points made in these seminal writings settled matters. For example, Paul Artin Boghossian, starts his 1996 article 'Analyticity Reconsidered' with the following remarks:

This is what many philosophers believe today about the analytic/synthetic distinction: In his classic early writings on analyticity — in particular, in "Truth by Convention," "Two Dogmas of Empiricism," and "Carnap and

Logical Truth” — Quine showed that there can be no distinction between sentences that are true purely by virtue of their meaning and those that are not. In so doing, Quine devastated the philosophical programs that depend on the notion of analyticity — specifically, the linguistic theory of necessary truth . . . Now, I do not know precisely how many philosophers believe all of the above, but I think it would be fair to say that it is the prevailing view.³³

Quine’s strategy against the analytic/synthetic distinction is stunningly novel and elegant. It targets its putative linguistic dimension through the syntax/semantics distinction. For Quine, if some propositions are true in virtue of linguistic conventions, then either their syntax or their semantics determines their truth. In “Two Dogmas,” he distinguishes between ‘*logically true*’ (syntactic) and others (semantic) analytic statements.³⁴ According to Quine, both the proof theoretical and model theoretical approaches to necessity can only account for analytic statements of the first kind. For the rest of the article, Quine attacks different attempts — mostly Carnap’s — at reducing analytic sentences of the second class to those of the first class. According to Quine, Carnap’s account of analyticity is unsuitable, because it tries to reduce semantics to syntax. For Quine, ‘analytic’ is an irreducible semantic notion. He finds no non-circular, suitable, semantic account of analyticity. For Quine, the usual attempts at semantically defining analyticity are circular, because they require a previous semantic understanding of analyticity.

Wittgenstein’s account of analyticity is not semantical, but syntactic. However, it does not correspond fully to Quine’s notion of logical truth. Quine’s definition of logical truths reformulates Yehoshua Bar-Hillel’s reconstruction of Bolzano’s definition of analytic proposition.³⁵

³³. *Nous* 1996.

³⁴. W. V. O. Quine, “Two Dogmas of Empiricism” *The Philosophical Review* 60, no. 1 (January 1951) 23.

³⁵. “If we suppose a prior inventory of *logical* particles, comprising ‘no’, ‘un-’, ‘not’, ‘if’, ‘then’, ‘and’, etc., then in general a logical truth is a statement which is true and remains true under all reinterpretations of its components other than the logical particles.” *Ibid.* 23

First, we suppose indicated, by enumeration if not otherwise, what words are to be called logical words; typical ones are ‘or’, ‘not’, ‘if’, ‘then’, ‘and’, ‘all’, ‘every’, ‘inly’, ‘some’. The logical truths, then, are those true sentences which involve only logical words *essentially*. What this means is that any other words, though they may also occur in a logical truth (as witness ‘Brutus’, ‘kill’, and ‘Caesar’ in ‘Brutus killed or did not kill Ceasar’), can be varied at will without engendering falsity.³⁶

As a matter of fact, Wittgenstein’s grammatical method is indeed very similar to one of the attempts at defining analyticity syntactically discussed in “Two Dogmas”. In section III, Quine discusses the account of analyticity, according to which (i) “any analytic statement could be turned into a logical truth by putting synonyms for synonyms”³⁷ and (ii), (cognitive) synonymy³⁸ is “interchangeability *salva veritate* everywhere except within words.”³⁹ According to him, this latter account is flawed, because interchangeability *salva veritate* does not capture cognitive synonymy, but only coextensionality. In consequence, not only analytic truths, but also synthetic truths may be transformed into logical truths through *salva veritate* substitution. For example, since the current president of Mexico in August 2000 is Ernesto Zedillo, the singular terms ‘current president of Mexico in August 2000’ and ‘Ernesto Zedillo’ are interchangeable *salva veritate*. In consequence, substituting ‘current president of Mexico in August 2000’ for ‘Ernesto Zedillo’ in the logical truth ‘The current president of Mexico in August 2000 is the current president of Mexico in August 2000’ results in the synthetic truth ‘The current president of Mexico in August 2000 is Ernesto Zedillo’. An endorser of this account may object that distinguishing between ‘current president of Mexico in August 2000’ and ‘Ernesto Zedillo’ remains possible. The terms cannot substitute for each other in a sentence like ‘Necessarily the

³⁶. (Quine 1963, 387)

³⁷. (Quine 1951, 28)

³⁸. Quine distinguishes cognitive analyticity from “synonymy in the sense of complete identity in psychological associations or poetic quality.” [p. 28]

³⁹. (Quine 1951, 28)

current president of Mexico in August 2000 is the current president of Mexico in August 2000', because 'Necessarily Ernesto Zedillo is the current president of Mexico in August 2000' is false. However, Quine retorts, this objection begs the question.

The above argument supposes we are working with a language rich enough to contain the adverb "necessarily", this adverb being so construed as to yield truth when and only when applied to an analytic statement. but can we condone a language which contains such an adverb? Does the adverb really make sense? To suppose that it does is to suppose that we have already made satisfactory sense of 'analytic'. Then what are we so hard at work on right now?⁴⁰

It is clear that Wittgenstein's grammatical method is very similar to that of Section III in "Two Dogmas". However, they are also significantly different, and these differences are strong enough to elude Quine's criticisms. First of all, Wittgenstein's interchangeability criterion is not *salva veritate*, but *salva grammaticality*. Second, it is not an attempt at defining general synonymy, but grammatical synonymy. In other words, it applies only to grammatical terms, not to all terms in general. Hence, it does not attempt to reduce genuine semantics to syntax – certainly a doomed enterprise. It attempts to give a synonymy criteria for those words whose grammar entirely determines their meaning.

Wittgenstein's distinction between grammatical and genuine propositions is similar to that between analytic and synthetic statements. However, Wittgenstein's distinction presumes nothing about its empirical nature, while Quine's primary concern is with the empirical dimension of the analytic/synthetic distinction. Wittgenstein's distinction between grammatical and genuine propositions is closer to the current distinction between syntax and semantics. Wittgenstein bases his distinction at the level of propositions on a distinction at the level of concepts and objects (as shown in chapter 2). For Wittgenstein, grammatical terms are those whose grammar entirely determines their meaning. Since grammatical

⁴⁰. (Quine 1951, 29)

concepts lack intensionality, co-extensionality offers suitable criteria for synonymy among grammatical terms.

Indeed, Wittgenstein never maintained that grammar fully determined the meaning of all terms. However, he argued that it did for those he called ‘grammatical’. In Wittgenstein’s grammatical method, grammatical concepts are grammatical categories, given by linguistic contexts. Two terms are grammatically equivalent if they are interchangeable *salva grammaticality* in all contexts. If the terms are grammatical, they are also synonymous. They have the same grammatical category as their meaning.

At the level of statements, a statement is grammatical if its concepts are grammatical concepts. In consequence, its grammar completely determines its ‘meaning’ and ‘truth’. In contrast, grammar cannot fully determine the truth of genuine propositions, but only their possibility. If a non-grammatical statement is well-formed, its meaning is a genuine proposition. It expresses a possible state of affairs. Modality is already built into the grammar of the language. In consequence, Wittgenstein’s grammatical account does not require a previous understanding of synonymy and, hence, is not circular in Quine’s sense.

2. Convention and Justification

But still there was no truth by convention,
because there was no truth.

Quine 1963, 392

The breadth of Quine’s arguments in “Truth by Convention” focuses on the foundational role of linguistic conventions. In consequence, it is mostly irrelevant for Wittgenstein’s grammatical project. Clearly, Wittgenstein found such a foundational enterprise absurd. His philosophy of mathematics during the middle period is not a conventionalism in that sense.

The target of Quine’s anti-conventionalist arguments is linguistic conventions’ inability to found mathematics or calculus. In other words, in “Truth by Convention”,

Quine questions linguistic conventions' capacity to justify mathematical or logical truths. However, Wittgenstein's grammatical account of mathematics is not a foundational enterprise. In Wittgenstein's account, grammatical rules certainly have no justificatory power. Wittgenstein most likely would sympathize with Quine's efforts to demonstrate the impossibility of justifying logical and mathematical truths by inferring them from syntactic conventions.

. . . the difficulty is that if logic is to proceed *mediately* from conventions, logic is needed for inferring logic from the conventions.⁴¹

Dummett reiterates this criticism when he says, in his "Wittgenstein on Necessity":

The moderate conventionalist view was never a solution to the problem of logical necessity at all, because, by invoking the notion of consequence, it appealed to what it ought to have been explaining: that is why it appears to call for a metanecessity beyond the necessity it purported to account for. The conventionalists were led astray by the example of the founders of modern logic into concentrating on the notion of logical or analytic *truth*, whereas precisely what they needed to fasten on was that of deductive *consequence*. .
.⁴²

Wittgenstein would agree with Quine and Dummett that logical truths and linguistic conventions do not entail each other logically. If conventions logically entailed logical truths, justifying this relation would itself require logic. 'Logical entailment' and 'justification' are concepts that do not apply to grammatical propositions, at least not in the same sense as they apply to genuine propositions.

If 'to justify p ' means to demonstrate the truth of p , then justification applies only to genuine propositions. Correct calculations are also called mathematical truths, but mathematical truth is not a sub-species of 'truth' in general. For Quine, "We may mark out the intended scope of the term 'logical truth', within that of the broader term 'truth'."⁴³ How-

41. W. V. O. Quine, *Mathematical Logic* (Cambridge: Harvard University Press, 1951) 97.

42. Michael Dummett, *The Seas of Language* (Oxford: Clarendon Press, 1993) 460.

43. Quine [1963] p. 386.

ever, since the *Tractatus*, the scopes of ‘truth’ and ‘logical truth’ do not overlap for Wittgenstein. In Ramsey’s words, “It is important to see that tautologies are not simply true propositions, though for many purposes they can be treated as true propositions.”⁴⁴

Ramsey presented Wittgenstein’s position very clearly in his ‘Foundations of Mathematics’ where he wrote:

The assimilation of tautologies and contradictions with true and false propositions respectively results from the fact that tautologies and contradictions can be taken as arguments to truth-functions just like ordinary propositions, and for determining the truth or falsity of the truth-function, tautologies and contradictions among its arguments must be counted as true and false respectively. Thus, if ‘*t*’ be a tautology, ‘*c*’ a contradiction, ‘*t* and *p*’, ‘If *t*, then *p*’, ‘*c* or *p*’ are the same as ‘*p*’, and ‘*t* or *p*’, ‘if *c*, then *p*’ are tautologies.⁴⁵

For Wittgenstein, ‘truth’ means something different when applied to tautologies than it does when applied to genuine propositions. In the logical calculus of propositions, being true is having ‘truth’ as truth value. In the truly semantic case, being true means that the proposition is the case. “For what does a proposition’s ‘*being true*’ mean? ‘*p* is true = *p*. (That is the answer.)”⁴⁶ In the case of tautologies and contradictions, nothing could or could not be the case. In consequence, saying that they are true (or false for that matter) in the same sense as true genuine propositions makes no sense. The predicate ‘true’, defined for genuine propositions, does not apply to tautologies or contradictions.

Mathematics is pure calculus, and every calculus is a rule-governed practice. In this respect, calculi are more like chess than like natural science. Asking for the justification of a mathematical truth is like asking for the justification of the truth of chess rules. Both are nonsense. It makes sense to justify ‘that *p*’, but not to justify ‘**to** *p*’. Unless justification

⁴⁴. Frank P. Ramsey, ‘The Foundations of Mathematics (1925)’ in *Philosophical Papers* (Cambridge: Cambridge University Press, 1990) 173.

⁴⁵. *Ibid.* 174.

⁴⁶. RFM Pt. I appendix I, §6

means something different when applied to rules and practices than to genuine propositions. For Carnap, a calculus is as ‘justified’ as its application. Carnap’s conventionalism is also a pragmatism. Application justifies calculation. Wittgenstein offers a different interpretation. For him, following a rule justifies it. A rule is justified if it is possible to follow it. This sense of justification does not require metamathematics. Performing the calculation is sufficient. It demonstrates that following the rule is constructively possible. Wittgenstein’s grammatical necessity is the necessity of calculations, not of propositions.⁴⁷

Finally, Wittgenstein is not a conventionalist in Dummett’s sense either. According to Dummett, Wittgenstein is a *radical* conventionalist, because he grounds mathematical necessity on the *decision* of not questioning mathematical truth. However, for the middle Wittgenstein, deciding whether or not to question mathematical propositions is absurd. Questioning grammatical propositions does not make sense. Accordingly, the mere notion of such a decision is nonsensical. Mathematical propositions are not the kind of things it makes sense to question. Hence, mathematics contains no decisions and, in consequence, no radical conventions, either.⁴⁸

IV. Conclusion: Wittgenstein’s Own Account of Analyticity

Wittgenstein’s grammatical account of the analyticity of internal descriptions in general, and mathematical propositions in particular, differs from Carnap and most recent accounts of analyticity, because it is not metaphysical or epistemological, but logical. In his 1996 article ‘Analyticity Reconsidered’, Paul Boghossian distinguishes between two different notions of analyticity: a metaphysical and an epistemological one.

⁴⁷. His interest is precisely what Dummett calls necessary consequence: what necessarily *follows* according to a rule.

⁴⁸. Do not confuse convention with stipulation. Grammatical rules may be conventions, but they are certainly not stipulations.

Here, it would seem, is one way: *If mere grasp of S's meaning by T sufficed for T's being justified in holding S true. . .* On this understanding, then, 'analyticity' is an overtly *epistemological* notion: a statement is 'true by virtue of its meaning' provided that grasp of its meaning alone suffices for justified belief in its truth.

Another, far more *metaphysical* reading of the phrase 'true by virtue of its meaning' is also available, however, according to which a statement is analytic provided that, in some appropriate sense, it *owes its truth value completely to its meaning*, and not at all to 'the facts'.

Wittgenstein's analyticity is neither metaphysical nor epistemological. Wittgenstein agrees with Kant that separating analyticity from aprioricity is important. 'Analyticity' is a logical notion, while 'apriori' is epistemological. However, Wittgenstein understands analyticity not as 'true by virtue of meaning', but 'true by virtue of grammar.'⁴⁹ Grammatical analyticity is not a semantic notion, but a logical one. Wittgenstein's account says that a statement **S** is analytic if and only if the mere inclusion of **S** in the language suffices for its truth. The term 'inclusion' in this characterization misleads, since the proposition does not exist outside **S**. Accordingly, the mere existence of **S** guarantees its truth. A grammatical statement **S** cannot exist and be false.

The truth of a mathematical calculus is not contingent on the existence of genuine objects, but only those mathematical ones it constructs for itself. No calculus requires the

⁴⁹. However, some scholars consider Wittgenstein's analyticity epistemological. Alberto Coffa ["Carnap, Tarski and the Search for Truth," *Nous* 21, no. 4 (December 1987) : 547-572] interprets Wittgenstein's account of analyticity – from the *Tractatus* to the middle and late periods of his philosophy – as epistemological. Wittgenstein characterizes logical sentences in the *Tractatus* as those "one can recognize [*erkennen*] from the symbol alone that they are true" [6.113] Coffa also recognizes "that this determination is embodied in constructive procedures that allow someone who understands the given language to 'recognize' the truth-values in question." [pp. 547, 548] Nevertheless, he does not interpret this procedure as a syntactic/grammatical one, but as an epistemic one. Michael Hymers ["Internal Relations and Analyticity: Wittgenstein and Quine" *Canadian Journal of Philosophy* 26, no. 4 (December 1996) : 591-612] also sustains that Wittgenstein's criteria for recognizing analytic propositions [internal descriptions] remained epistemological from the *Tractatus* to PG. [p. 594] He writes, "Also implicit here [in the *Philosophical Grammar*], is a further revision of the epistemic criterion for internal relations: two concepts, or instruments of language, are internally related if in order to understand one I must also understand the other. . . However, concepts have no existence here, independently of norms and practices. Understanding a concept is, paradigmatically, to be able to use a word correctly, where correctness amounts to accord with the rules of a calculus." [pp. 596-597]

existence of any spatio-temporal objects or events. For example, arithmetical addition is not contingent on any particular numerals, or additions. A mathematical statement like ‘ $3 + 4 = 7$ ’ says that the correct result of adding three to four is seven. However it does not refer to any particular numerals or additions. The equation refers to numbers as *roles* in the calculus and to additions as calculations: entities fully defined by the calculus’ rules.

Mathematics is part of the syntax of language. However, mathematics is not describing this syntax in a metalanguage. Describing syntax is substantially different than calculating. A meta-linguistic description of logical syntax, like Carnap’s, is external, while mathematics is autonomous. The mere description of a calculus’ external *Anwendung* cannot fully determine the correctness or incorrectness of its propositions.

Mathematics is pure calculus, and mathematical propositions are calculation rules. ‘Justification’ and ‘truth’ apply to genuine propositions only. They do not apply to rules. Justifying a mathematical truth is as absurd an enterprise as justifying the truth of a chess rule, for example. Grammatical necessity is the necessity of calculations, not of propositions. It requires no further justification. Performing the calculation is enough to guarantee its ‘truth’, because a calculation is autonomous. It cannot exist as a calculation and be false.

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